8^{th} Grade Mathematics

Unit 7: Transformations

Worth County Middle School
2015-2016

8th Grade Math Unit 7: Transformations, *Congruence*, & Similarity

Unit EQ: How can I understand congruence and similarity using physical models, transparencies, or appropriate technology?

Transformations	*Congruence & Similarity*	*Angle Relationships*
Essential Questions: • How can the coordinate plane help me understand properties of reflections, translations, and rotations? • What is the relationship between reflections, translations, and rotations? • What is a dilation and how does this transformation affect a figure in the coordinate plane?	**Essential Questions:** • How can I tell if two figures are similar? • In what ways can I represent the relationships that exist between similar figures using the scale factors, length ratios, and area ratios? • What strategies can I use to determine missing side lengths and areas of similar figures? • Under what conditions are similar figures congruent?	**Essential Questions:** • When I draw a transversal through parallel lines, what are the special angle and segment relationships that occur? • Why do I always get a special angle relationship when any two lines intersect? • How can I be certain whether lines are parallel, perpendicular, or skew lines?
Georgia Standards of Excellence: *MGSE8.G.1, MGSE8.G.2, MGSE8.G.3 MGSE8.G.4*	***Georgia Standards of Excellence:*** *MGSE8.G.1, MGSE8.G.2, MGSE8.G.4*	***Georgia Standards of Excellence:*** *MGSE8.G.1, MGSE8.G.5*
Vocabulary: Rotation Reflection Translation Dilation Angle of Rotation Dilation Reflection Reflection Line Rotation	**Vocabulary:** *Similar Figures* *Congruent Figures* *Corresponding Sides* *Corresponding Angles* *Scale Factor*	**Vocabulary:** *Same-Side Interior Angles* *Same-Side Exterior Angles* *Transversal* *Alternate Exterior Angles* *Alternate Interior Angles* *Linear Pair*

STANDARDS FOR MATHEMATICAL CONTENT

Understand congruence and similarity using physical models, transparencies, or geometry software.

MGSE8.G.1 Verify experimentally the properties of rotations, reflections, and translations: lines are taken to lines and line segments to line segments of the same length; angles are taken to angles of the same measure; parallel lines are taken to parallel lines.

MGSE8.G.2 Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.

MGSE8.G.3 Describe the effect of dilations, translations, rotations, and reflections on two- dimensional figures using coordinates.

MGSE8.G.4 Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them.

MGSE8.G.5 Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle- angle criterion for similarity of triangles. For example, arrange three copies of the same triangle so that the three angles appear to form a line, and give an argument in terms of transversals why this is so.

2.0 Transformation Basics

Transformations are when we take a picture or shape and change it using four different methods of sliding, rotating, reflecting, or shrinking/enlarging. We refer to the picture before the change as the **pre-image** and may give it a name such as A. We might also name the points that make up the picture the same way.

The picture after we apply the transformation is called the **image**. The name of the image is based on the name of pre-image. For example, if the pre-image were named A, then the image would be name A'. That would be pronounced, "A prime." We add the prime mark, really just a tick mark like an apostrophe, to show that it is the image, or picture after the transformation happened.

Some transformations keep the pre-image and image congruent. **Congruent** means that they are the same size and shape or that they have the same measurements. They make not have the same orientation, meaning they could be flipped or turned, but they do have the same measurements. Most transformations lead to congruent shapes.

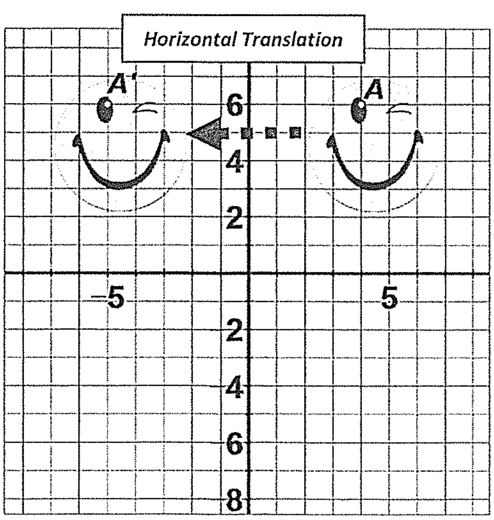
One transformation makes the image similar to the pre-image. **Similar** means that they are the same shape and proportion, but not the same size. In other words, the image was either shrunk or enlarged since it's not the same size. One of our main jobs throughout this unit is to make sure we know the difference between congruent and similar transformations.

Transformations Leading to Congruent Shapes

Three of the four transformations preserve the size and shape of the pre-image: translations, rotations, and reflections. In other words, if you translate a pre-image square that has side lengths of $4\ cm$, it will still have side lengths of $4\ cm$ after it is slid to its new position.

Translations

When we slide a picture or shape, it is called a **translation**. There are two directions we can slide when our pre-image is in a plane: vertically or horizontally. Even diagonal translations are combinations of vertical and horizontal movement. The following transformation is a horizontal translation to the left.



Note that the pre-image, the original picture, is labeled as A and is the smiley face in quadrant one. To create this transformation we just slid the smiley face to the left nine places. We'll talk about how to write that out mathematically a little bit later. For right, we just need to be able to recognize that this transformation is a translation.

Next we'll look at an example of a translation diagonally.

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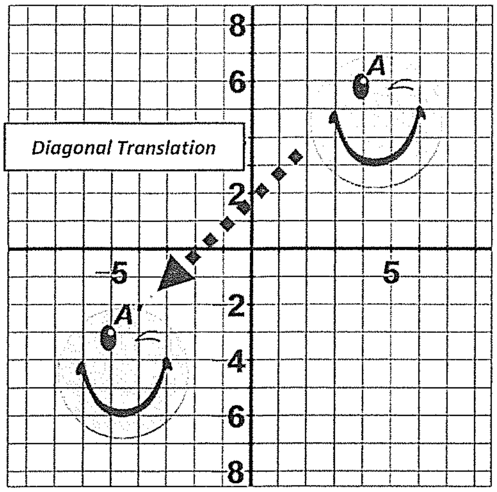
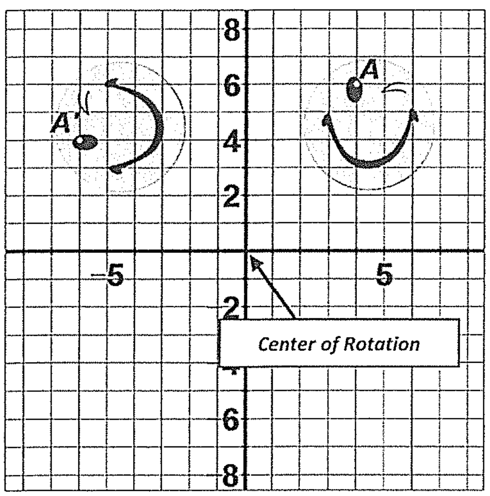


Note that this translation is really two parts: a horizontal translation left and a vertical translation down. What this shows us is that we can translate in any direction we want. If we need a diagonal translation we simply put together a horizontal and vertical part to get the diagonal we want.

Rotations

When we rotate a picture or shape by a specific angle measurement about a specific point, it is called a **rotation**. Rotations always occur in a counter-clockwise fashion. For our purposes, we will only rotate in 90° increments and we will usually rotate about the origin. Here's an example of rotating our smiley about the origin:



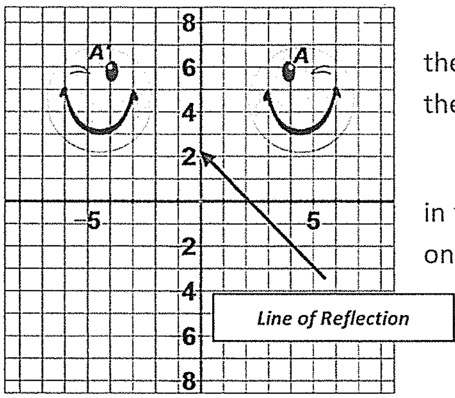
We can see that by rotating a figure, we did not change its size (so the smiley faces are still congruent), but the orientation did change. The rotated smiley face is on its side at this point.

What angle measurement would keep the orientation the same? If we did a 360° rotation, the image would be exactly the pre-image. They would be sitting right on top of each other. Because of this, we typically don't talk about a rotation of 360°.

The only rotations we will use will be 90° (a quarter turn), 180° (a half turn), and 270° (a three-quarters turn). Again, at this point we just need to be able to recognize a rotation.

Reflections

When we reflect a picture or shape across a line, it is called a **reflection**. Mathematically what is happening is every point on the pre-image is sent to a corresponding point on the other side of the line so that the distance from each point to the line is the same. Also, if you were to draw line between the point in the pre-image and its image, that line would be perpendicular to the reflection line. Here's an example of a reflection across the y-axis:



Notice that with the reflection the orientation changed (the wink is on the left in the image instead of the right like the pre-image), but the size stayed the same again maintaining congruence.

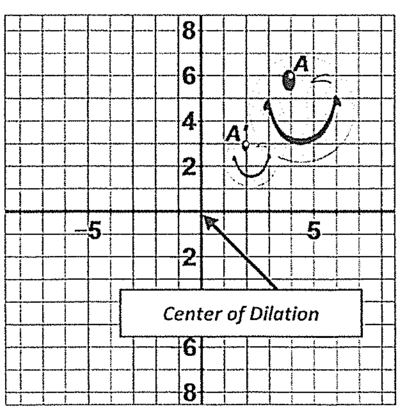
We could also reflect across the x-axis which would have put the image in the quadrant four (below the original smiley face). The two axes will be the only lines we will reflect across at the 8th grade level.

Transformations Leading to Similar Shapes

The fourth transformation does not preserve the size of the pre-image, only the shape. This transformation is called a dilation.

Dilations

When we shrink or enlarge a picture or shape, it is called a **dilation**. We need a scale factor to dilate by so we know how much to shrink or enlarge. The scale factor is the ratio of the size of the image to the size of the pre-image. So a scale factor of $\frac{1}{2}$ represents $\frac{1\ unit\ on\ image}{2\ units\ on\ pre-image}$, which is a shrinking scale since the pre-image is bigger. An enlarging scale would be a number greater than one, and a shrinking scale would be numbers between zero and one.



Besides a scale factor, we also need a center of dilation. The center of dilation is where things get compressed towards if we are shrinking or where things get stretched away from if we are enlarging. Again, for the ease of the math involved, we will only use the origin as our center of dilation. Here is an example of a shrinking dilation:

Notice that the smiley face was also pulled toward the origin during the shrinking process. Also notice that in this case the pre-image and image are the same shape but not the same size. This makes them similar instead of congruent.

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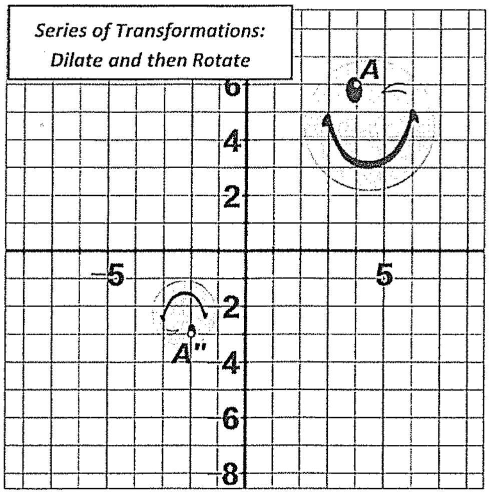
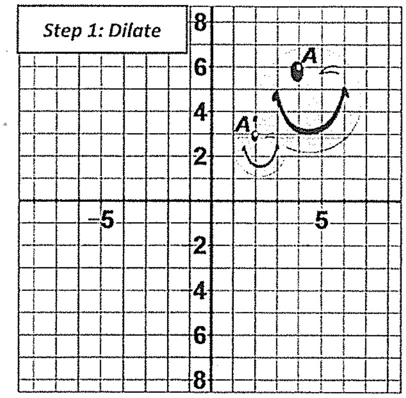
Combining Transformations

Finally, we can combine multiple transformations into a series. For example, we could dilate by half (shrink it) and then rotate by 180° as follows:

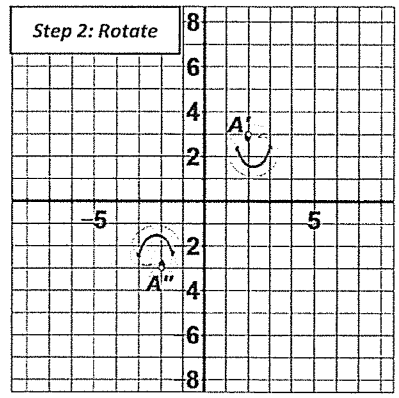


One thing to note is that this image is labeled as A'' instead of A'. This is because there were two transformations applied to the pre-image A to get the image.



Let's walk through the process. First, the image is dilated by half. That gives us the picture on the right.

The second step in the process is to rotate A' by 180°. In other words, we're taking the new picture we just created and rotating it. We're not rotating the original picture. You can see this step to the right again.



Another interesting thing to note about our series is that we could get the same image in another way. We could dilate by half, reflect across the x-axis, and then reflect across the y-axis. This highlights the fact that there may be more than one solution when deciding what transformations occurred in a series.

2.1 Constructing Dilations

It is useful to be able to perform transformations using the coordinate plane. This allows us to specify the exact coordinates of the pre-image, image, and other important reference points.

To help us do so, let's review our notation. Remember that a capital letter, such as A, represents the pre-image which is usually a point or a whole picture. In the following section, it will most often refer to a point. The prime following the capital letter, such as A', means the image. So we might say that A' is the image of A. Another way to think about it is with the words "new" and "old" so that A is the old point and A' is the new point.

We'll also apply this same notation to x- and y-coordinates so that x means the x-coordinate of the old point (pre-image) and x' means the new x-coordinate of the new point (image). Similarly y is the y-coordinate of the old point (pre-image) and y' means the new y-coordinate of the new point (image).

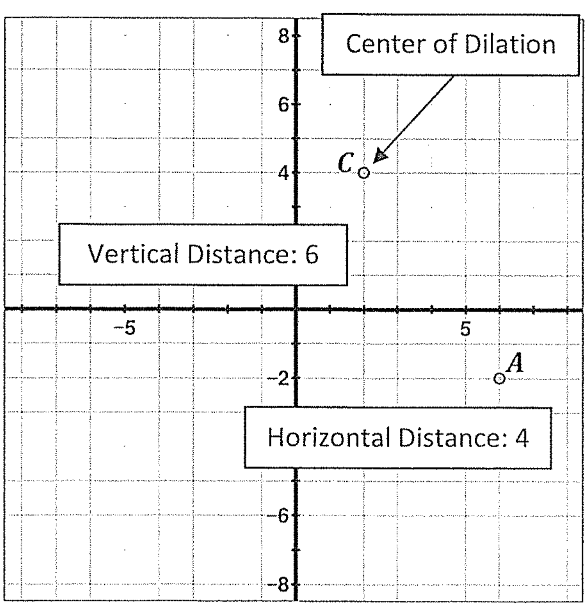
Also remember that dilation is the only transformation that creates similar but not congruent figures because of the shrinking or enlarging. This means that any time we dilate we are dealing with similarity.

Dilations

We focus on dilations first because it is usually best to perform this operation first in a series since dilating also pulls towards or away from the **center of dilation** which can affect later translations, rotations, or reflections. The center of dilation is the point on the coordinate plane that we are shrinking towards or enlarging away from.

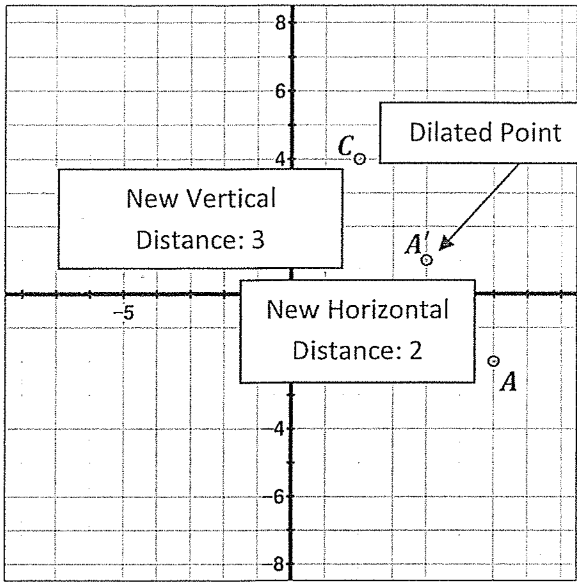
Beyond the center of dilation, we also need a **scale factor** which tells us by what factor we moving away from the center of dilation. So if we have a scale factor of $c = \frac{1}{2}$, then the image points will be half as far away from the center of dilation as the pre-image points were. That means the image will shrink. On the other hand, if we had a scale factor of $c = 3$, then the image would be three times farther away from the center of dilation than the pre-image points were. That would mean we were enlarging.

To dilate, we start at the center of dilation and count both the horizontal and vertical distance to get to the point we want to actually dilate. For example, consider dilating the point A: $(6, -2)$ by a scale factor of $c = \frac{1}{2}$ using a center of dilation of C: $(2,4)$. Notice that this will shrink the point, or pull it closer to the center of dilation since it will only be half as far away.



We get the vertical distance by starting at our center of dilation, point C, and counting down towards the pre-image point A until we are directly across from it. The vertical distance from the center of dilation to the point A is six units.

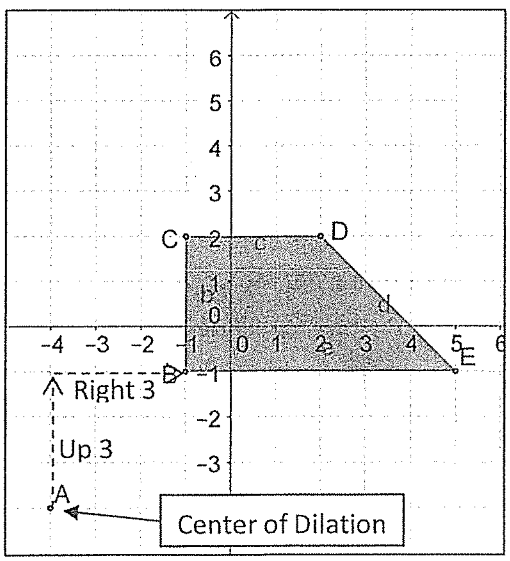
To get the horizontal distance, we now count from that new place over to point A. The horizontal distance between the center of dilation and the point A is four units.



Using this information we can plot the new point A' by taking each of those distances and multiplying by the scale factor. Since point A was six units down, point A' should be $\frac{1}{2} * 6 = 3$ or three units down. Then point A was four units right, so point A' should be $\frac{1}{2} * 4 = 2$ or two unit right. Overall that means we start at the center of dilation and move down three and then right two to find that the image point A' should be at $(4, 1)$.

That example was just dilating a single pre-image point, but dilating a full shape works the same way.

Dilating a full shape

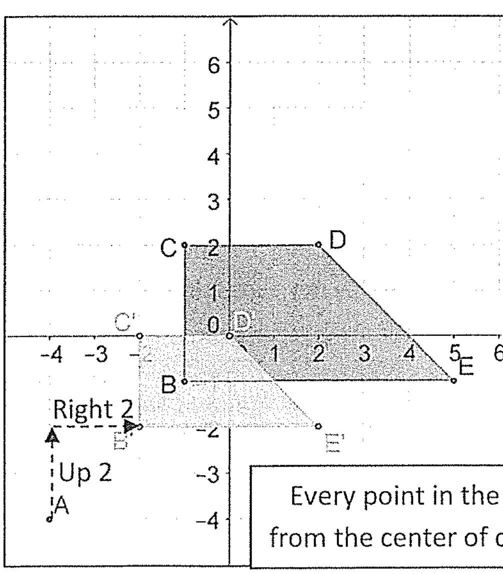


Let's start with the Trapezoid $BCDE$ and dilate it from a center of dilation at A: $(-4, -4)$ by a scale factor of $c = \frac{2}{3}$ meaning it will shrink and be $\frac{2}{3}$ as far from the center of dilation. Remember that we start by counting from the center of dilation to each point. Counting from A to B we see that it is three up and three right. Verify for yourself that the following distances are correct.

C is six up and three right from A
D is six up and six right from A
E is three up and nine right from A

That means to get to the new points of the trapezoid in the image, we'll need to multiply all of those distances by the scale factor of $c = \frac{2}{3}$. That should give us the following distances from the center of dilation. Remember to count from the center of dilation, not from the points themselves!

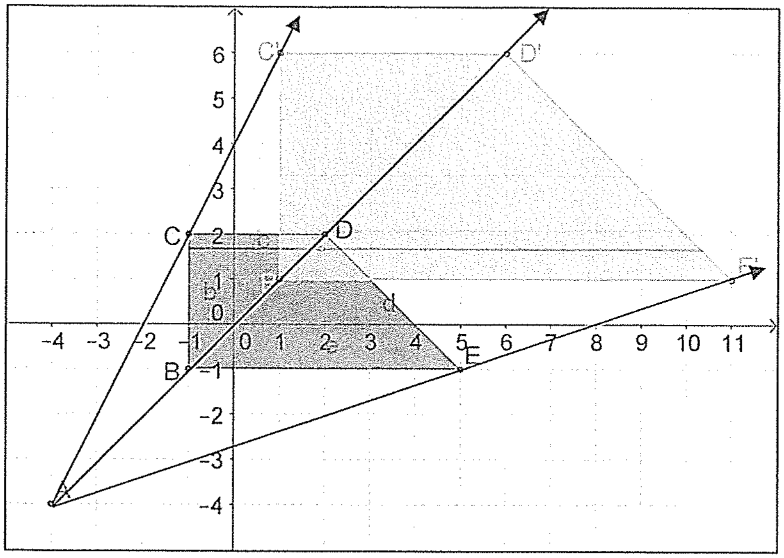
B' will be $\frac{2}{3} * 3 = 2$ or two up and $\frac{2}{3} * 3 = 2$ or two right from A
C' will be $\frac{2}{3} * 6 = 4$ or four up and $\frac{2}{3} * 3 = 2$ or two right from A
D' will be $\frac{2}{3} * 6 = 4$ or four up and $\frac{2}{3} * 6 = 4$ or four right from A
E' will be $\frac{2}{3} * 3 = 2$ or two up and $\frac{2}{3} * 9 = 6$ or six right from A



So we will move those distances from the center of dilation to plot all the new points in the image. Once we connect all of those points, we see the new trapezoid as shown in green to the left.

Every point in the image is two-thirds as far from the center of dilation as in the pre-image.

Enlarging the trapezoid



If we take that same trapezoid and center of dilation and dilate it by a scale factor of $c = \frac{5}{3}$, we'll get the following distances.

B' will be $\frac{5}{3} * 3 = 5$ or five up and $\frac{2}{3} * 5 = 5$ or five right from A

C' will be $\frac{5}{3} * 6 = 10$ or ten up and $\frac{5}{3} * 3 = 5$ or five right from A

D' will be $\frac{5}{3} * 6 = 10$ or ten up and $\frac{5}{3} * 6 = 10$ or ten right from A

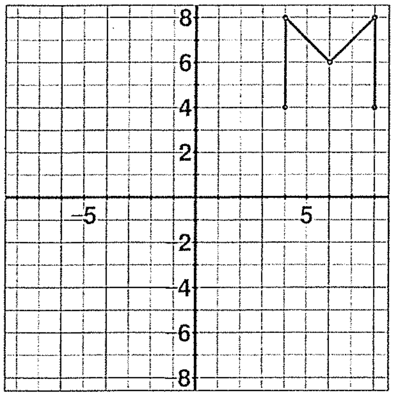
E' will be $\frac{5}{3} * 3 = 5$ or five up and $\frac{5}{3} * 9 = 15$ or fifteen right from A

Here you can see the image trapezoid $B'C'D'E'$ is $\frac{5}{3}$ times as far away from the center of dilation as it was in the pre-image. You can even think of connecting the center of dilation to each of the points in the pre-image with a ray (arrow only in one direction) and see that they hit the corresponding points in the image. It's a like a funnel that the pre-image will either shrink or enlarge within.

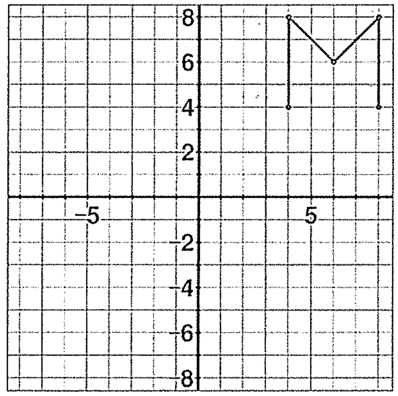
Lesson 2.1

Perform the given dilation on each given pre-image with the given center of dilation.

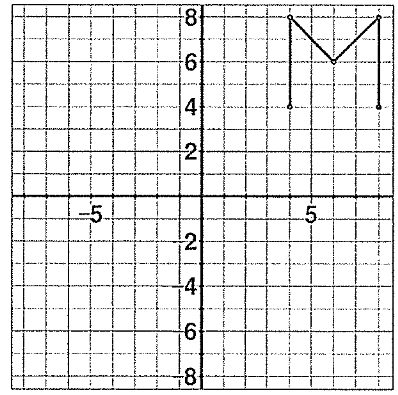
1. Dilate by $c = \frac{1}{4}$, center (0,0)



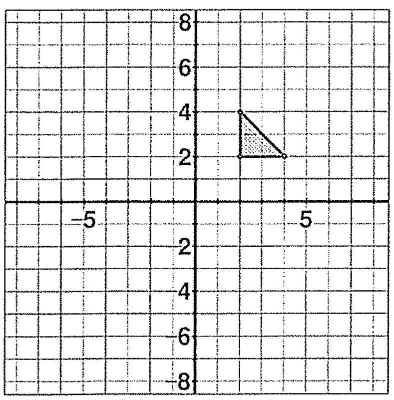
2. Dilate by $c = \frac{1}{2}$, center (2,2)



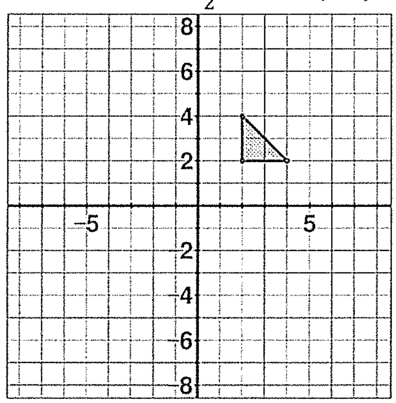
3. Dilate by $c = \frac{3}{4}$, center (0,0)



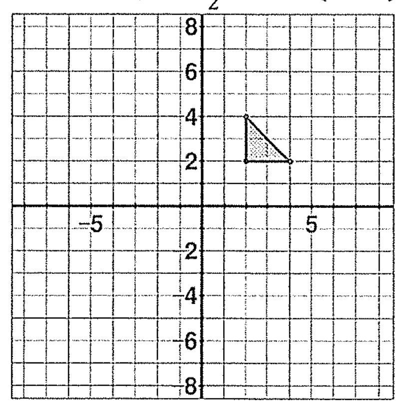
4. Dilate by $c = 2$, center (6,4)



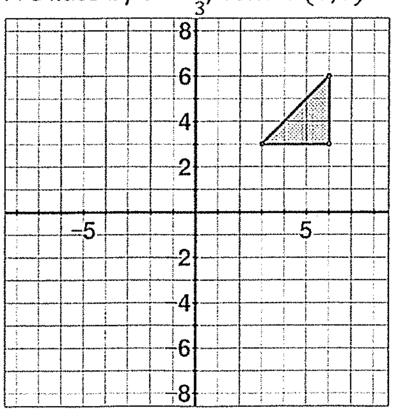
5. Dilate by $c = \frac{3}{2}$, center (0,0)



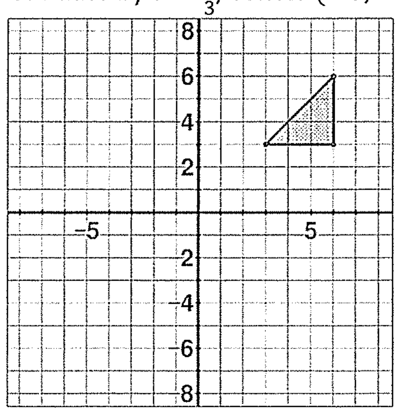
6. Dilate by $c = \frac{1}{2}$, center (−6,2)



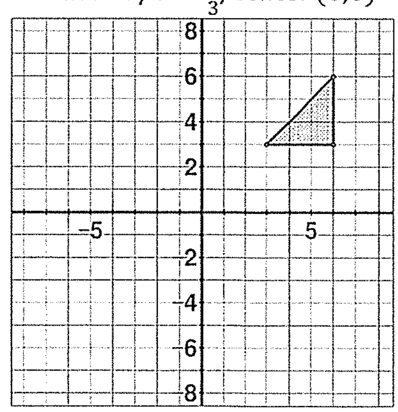
7. Dilate by $c = \frac{1}{3}$, center (0,0)



8. Dilate by $c = \frac{2}{3}$, center (−3, −6)



9. Dilate by $c = \frac{4}{3}$, center (0,0)



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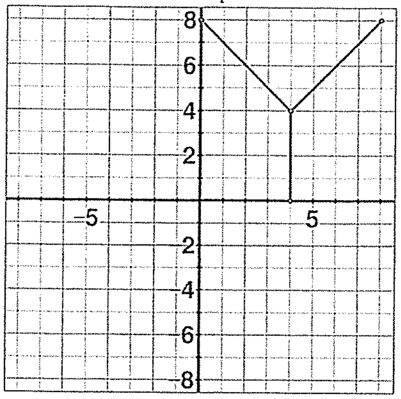
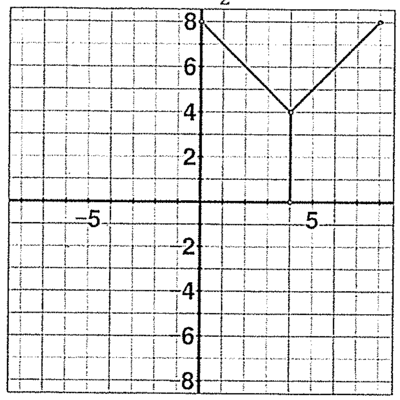
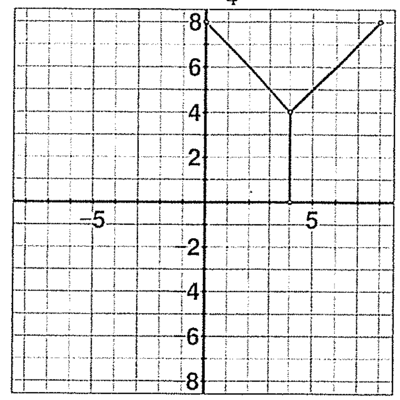
10. Dilate by $c = \frac{1}{4}$, center (4,4)



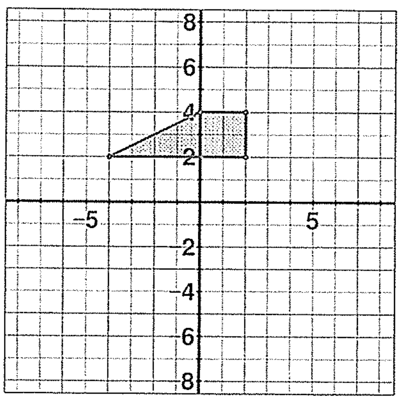
11. Dilate by $c = \frac{1}{2}$, center (0,0)



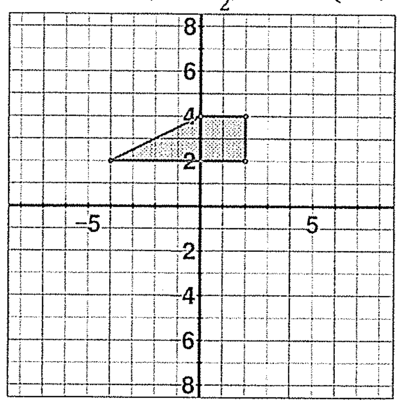
12. Dilate by $c = \frac{3}{4}$, center (−4,8)



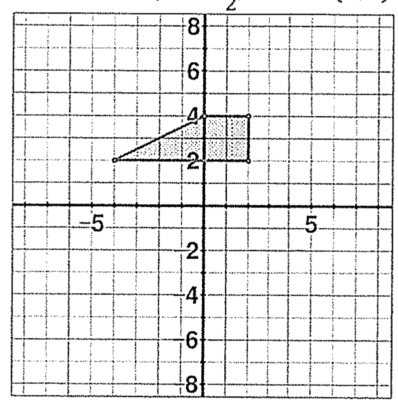
13. Dilate by $c = 2$, center (0,0)



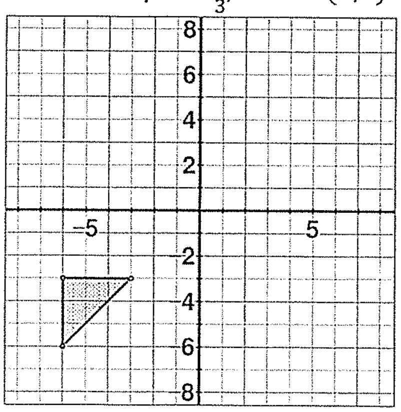
14. Dilate by $c = \frac{3}{2}$, center (−4,−2)



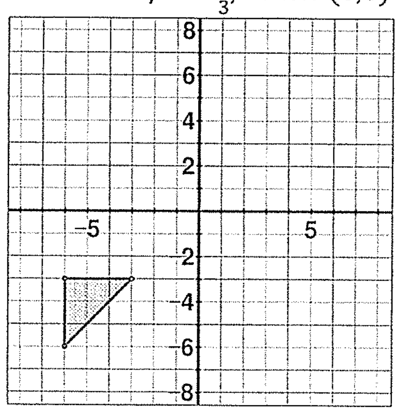
15. Dilate by $c = \frac{1}{2}$, center (0,0)



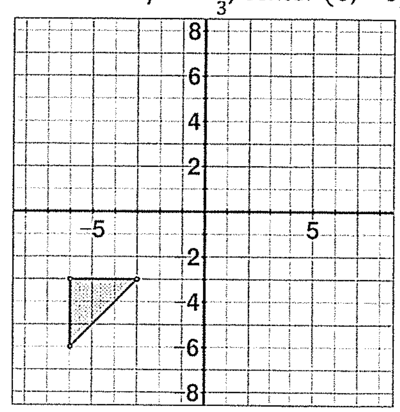
16. Dilate by $c = \frac{1}{3}$, center (3,0)



17. Dilate by $c = \frac{2}{3}$, center (0,0)



18. Dilate by $c = \frac{4}{3}$, center (0, −6)



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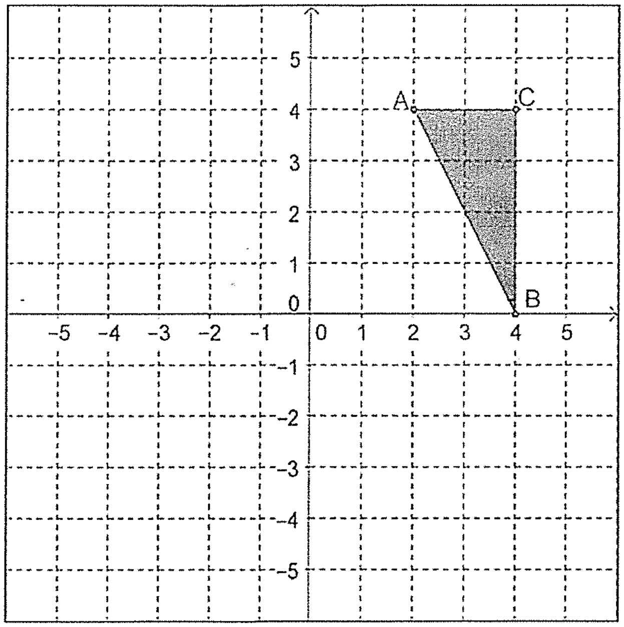


19. Dilate the triangle by a scale factor $c = \frac{1}{2}$ using the origin as the center of dilation. Which of the following statements will be true? For each explain why or why not.

a. Point A' is at the same coordinates as Point A.

b. Point B' is at the same coordinates as Point B.

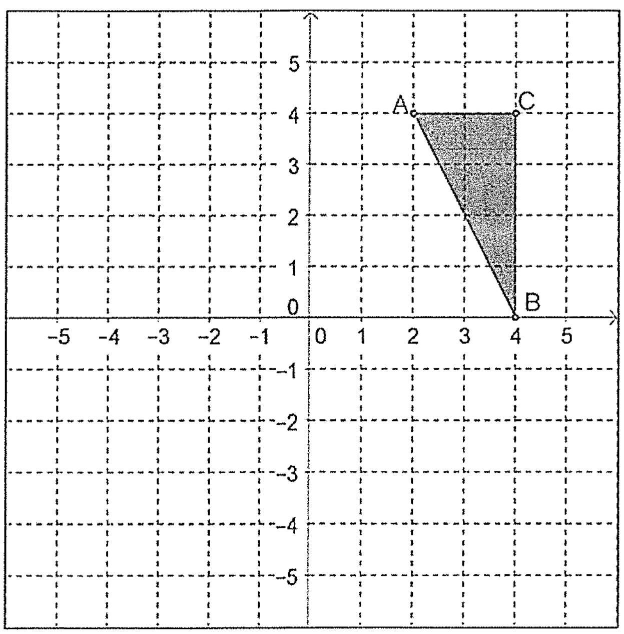
c. Point C' is at the same coordinates as Point C.

d. The image's perimeter equals the pre-image's.

e. The image's area equals the pre-image's.

f. The image is congruent to pre-image.

g. Line segment $\overline{A'C'}$ is horizontal.



20. Dilate the triangle by a scale factor $c = 2$ using the point (4,4) as the center of dilation. Which of the following statements will be true? For each explain why or why not.

a. Point A' will be at the same coordinates as Point A.

b. Point B' will be at the same coordinates as Point B.

c. Point C' will be at the same coordinates as Point C.

d. The image's perimeter equals the pre-image's.

e. The image's area equals the pre-image's.

f. The image is similar to pre-image.

g. Line segment $\overline{B'C'}$ is vertical.

21. In your own words, explain what a dilation does to a pre-image. Remember to consider the center of dilation in your explanation.

22. How could dilations be used in real life?

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Reteach

Dilations

A **dilation** is a transformation that enlarges or reduces a figure by a scale factor. The preimage and image are similar figures.

Example 1

A triangle has vertices $C(-2, -1)$, $D(1, 1)$, and $E(2, -3)$. Find the coordinates of the vertices of the triangle after a dilation with a scale factor of 2.

The dilation is $(x, y) \rightarrow (2x, 2y)$. Multiply the coordinates of each vertex by 2.

$C(-2, -1) \rightarrow [2 \cdot (-2), 2 \cdot (-1)] \rightarrow (-4, -2)$

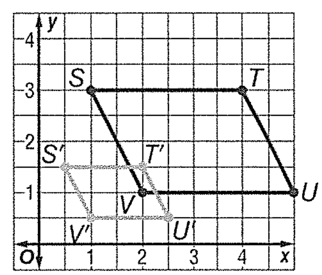
$D(1, 1) \rightarrow (2 \cdot 1, 2 \cdot 1) \rightarrow (2, 2)$

$E(2, -3) \rightarrow [2 \cdot 2, 2 \cdot (-3)] \rightarrow (4, -6)$

So, the coordinates after the dilation are $C'(-4, -2)$, $D'(2, 2)$, $E'(4, -6)$.

Example 2

A figure has vertices $S(1, 3)$, $T(4, 3)$, $U(5, 1)$, and $V(2, 1)$. Graph the figure and its image after a dilation with a scale factor of $\frac{1}{2}$.



The dilation is $(x, y) \rightarrow \left(\frac{1}{2}x, \frac{1}{2}y\right)$. Multiply the coordinates of each vertex by $\frac{1}{2}$.

$S(1, 3) \rightarrow \left(\frac{1}{2} \cdot 1, \frac{1}{2} \cdot 3\right) \rightarrow \left(\frac{1}{2}, 1\frac{1}{2}\right)$

$T(4, 3) \rightarrow \left(\frac{1}{2} \cdot 4, \frac{1}{2} \cdot 3\right) \rightarrow \left(2, 1\frac{1}{2}\right)$

$U(5, 1) \rightarrow \left(\frac{1}{2} \cdot 5, \frac{1}{2} \cdot 1\right) \rightarrow \left(2\frac{1}{2}, \frac{1}{2}\right)$

$V(2, 1) \rightarrow \left(\frac{1}{2} \cdot 2, \frac{1}{2} \cdot 1\right) \rightarrow \left(1, \frac{1}{2}\right)$

Example 3

BIKES Zach has a toy bicycle that is 2 inches tall. An actual tricycle that is 16 inches tall. What is the scale factor of the dilation?

Write a ratio comparing the heights of the two bicycles.

$$\frac{\text{height of actual}}{\text{height of toy}} = \frac{16}{2} = 8$$

So, the scale factor of the dilation is 8.

Exercises

Find the coordinates of the vertices of the figure after a dilation with the given scale factor k. Then graph the original image and the dilation.

1. $W(-1, 4)$, $X(1, 2)$, $Y(-2, 1)$; $k = 3$

2. BOXES Janelle has a box that measures 4 feet by 3 feet. She needs a box similar to this one that measures 2 feet by 1.5 feet. What is the scale factor of the dilation?

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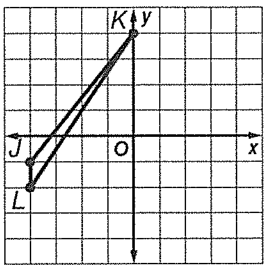
Skills Practice

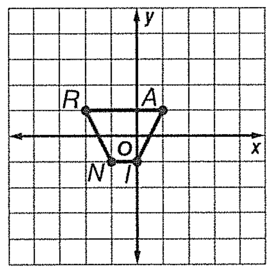
Dilations

Find the coordinates of the vertices of each figure after a dilation with the given scale factor k. Then graph the original image and the dilation.

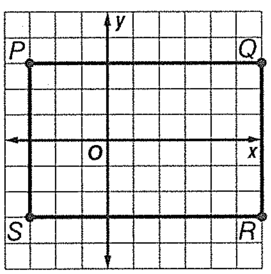
1. $J(-4, -1), K(0, 4), L(-4, -2); k = \frac{1}{2}$



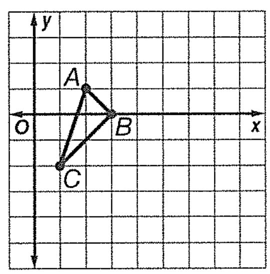
2. $R(-2, 1), A(1, 1), I(0, -1), N(-1, -1); k = 2$



3. $P(-3, 3), Q(6, 3), R(6, -3), S(-3, -3);$ $k = \frac{1}{3}$



4. $A(1, -2), B(2, 1), C(3, 0); k = 3$



5. PHOTOS Kiesha used a photo that measured 4 inches by 6 inches to make a copy that measured 8 inches by 12 inches. What is the scale factor of the dilation?

6. MODELS David built a model of a regulation basketball court. His model measured approximately 3.75 feet long by 2 feet wide. The dimensions of a regulation court are 94 feet long by 50 feet wide. What is the scale factor David used to build his model?

7. BLUEPRINTS On the blueprints of Mr. Wong's house, his great room measures 4.5 inches by 5 inches. The actual great room measures 18 feet by 20 feet. What is the scale factor of the dilation?

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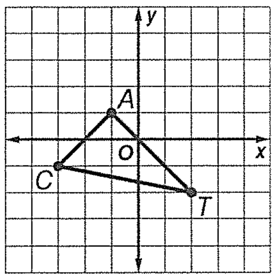
Problem-Solving Practice

Dilations

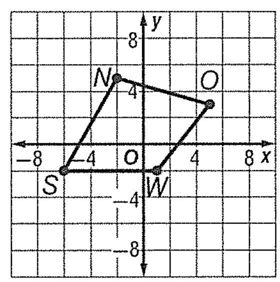
1. **GEOMETRY** Find the coordinates of the triangle shown below after a dilation with a scale factor of 4.



2. **PHOTOS** Daniel is using a scale factor of 10 to enlarge a class photo that measures 3.5 inches by 5 inches. What are the dimensions of the photo after the dilation?

3. **DOGS** Isabel has a mother dog and her puppy that look exactly alike. The puppy weighs 6 pounds, and the mother weighs 48 pounds. Assuming the two dogs are similar, what is the scale factor of the dilation?

4. **GEOMETRY** Find the coordinates of the quadrilateral shown below after a dilation with a scale factor of $\frac{1}{2}$.



5. **BLUEPRINTS** Abby's family is building a new house. On the blueprints of the house, Abby's bedroom measures 3 inches by 3.75 inches. Her actual bedroom will measure 8 feet by 10 feet. What is the scale factor for the dilation?

6. **ART** William saw a painting in a museum, and later found a picture of that same painting in a book. The actual painting measured 36 inches by 54 inches. The picture of the painting measured 4 inches by 6 inches. What is the scale factor for the dilation?

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2.2 Constructing Reflections

Now we begin to look at transformations that yield congruent images. We'll begin with reflections and then move into a **series** of transformations. A series of transformations applies more than one transformation one at a time to a pre-image.

Reflections

Reflections are like a mirror image, or flip, of the pre-image. This results in a congruent image, meaning it is not only the same shape but also the same size. At the 8th grade, we will usually be reflecting across the x-axis or the y-axis. Both cases need different formulas for the coordinates, so we'll break it down one at a time.

Reflecting Across the x-Axis

A reflection across the x-axis takes points above the x-axis to below the x-axis and vice versa. It basically flips the shape up and down. To do so, it utilizes these formulas:

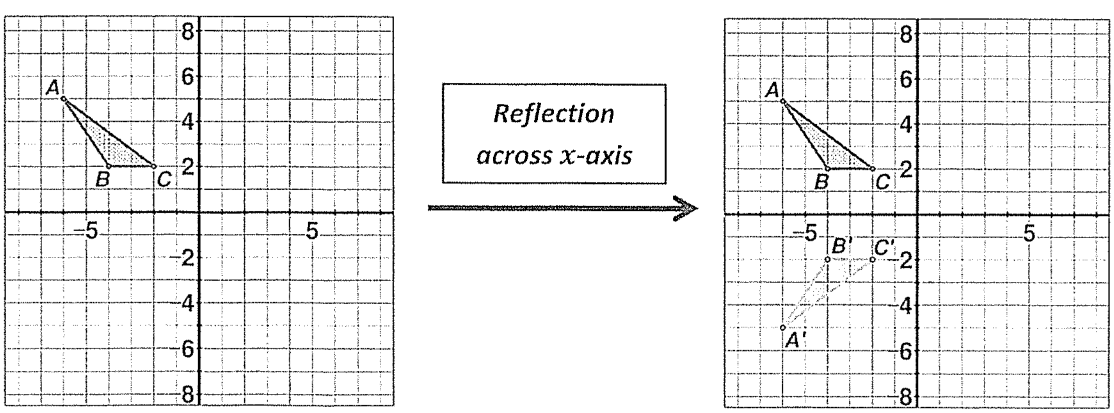
$$x' = x$$

$$y' = -y$$

Notice that the x-coordinate stays the same in the image as it was in the pre-image. The only real change is that you take the opposite y value. So if the y-coordinate was positive in the pre-image, it would be negative in the image. If it was negative in the pre-image, it would be positive in the image. Basically, if you want to apply a reflection across the x-axis, just change the sign of the y-coordinates.

While the formulas are valid for this reflection, you can imagine how annoying it would be to memorize the formulas for reflections across lots of different lines. Therefore, it is probably much simpler for you to think of a reflection as folding the paper on the line of reflection and stamping the pre-image wherever it lands after the fold.

Let's take a triangle with coordinates $A: (-6,5), B: (-4,2), C: (-2,2)$ and reflect it across the x-axis. Since we only need to change the sign of the y-coordinates we end up with the image of a triangle with the points $A': (-6,-5), B': (-4,-2), C': (-2,-2)$ if we use the formulas. If you simply think of folding the paper on the x-axis, notice that the blue triangle would land exactly where the green one is. It's like the blue triangle is a stamp and stamped down the new green triangle.



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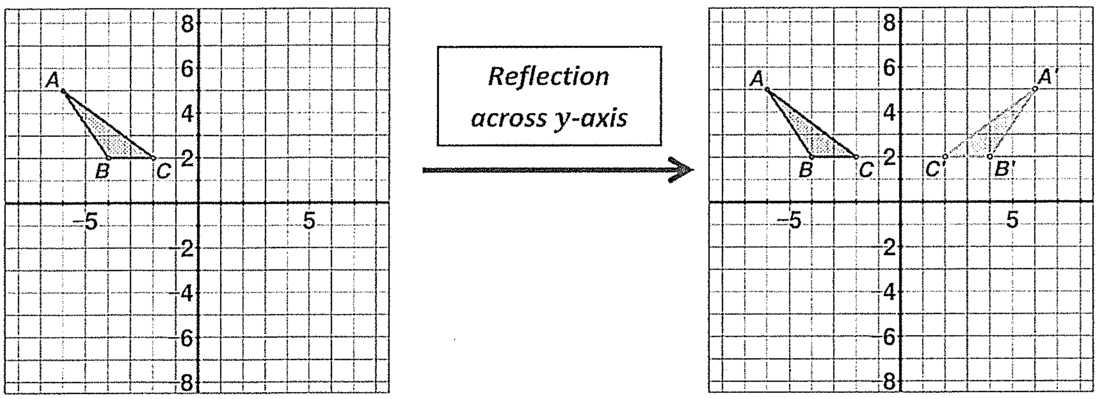
Reflecting Across the y-Axis

Seeing how to reflect across the x-axis, what do you think will happen if you reflect across the y-axis? As expected, it is now the x-coordinates that change, and the y-coordinates that stay the same. We use these formulas to reflect across the y-axis:

$$x' = -x$$

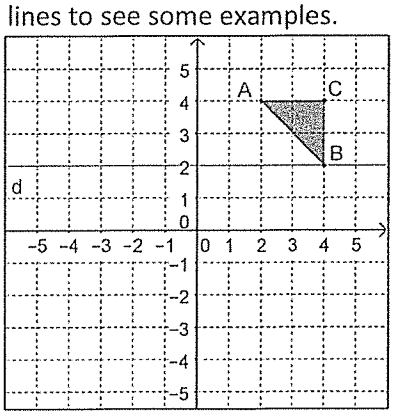
$$y' = y$$

Just change the sign of the x-coordinate. Let's look at the triangle from our previous example and reflect it across the y-axis. You should verify that it leads to the points A': (6,5), B': (4,2), C: (2,2). However, notice that it would again be easier to think of folding the paper on the y-axis and stamping the green triangle from the blue.

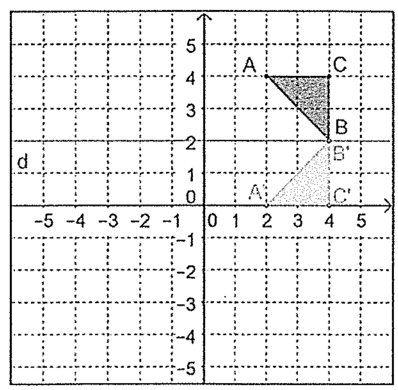


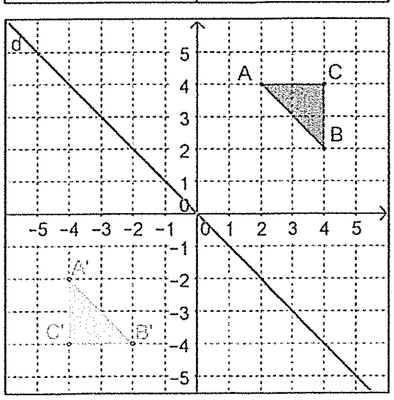
Reflecting Across Other Lines

Since every line has a new formula to reflecting the coordinates, we'll stop looking at those equations. Instead, we'll focus only on the "fold and stamp" idea. Let's take a triangle and reflect across a couple of different lines to see some examples.



We'll begin by reflecting the triangle across the line $y = 2$ which is a horizontal line at a height of 2 as seen to the left. Imagine folding the paper on that red line. Where would the blue triangle land? Stamp a green triangle there in your mind. Did you get the picture to the right?





Now let's reflect that same triangle across a diagonal line of $y = -x$. Fold and stamp and you should get the picture to the left. Fold and stamp!

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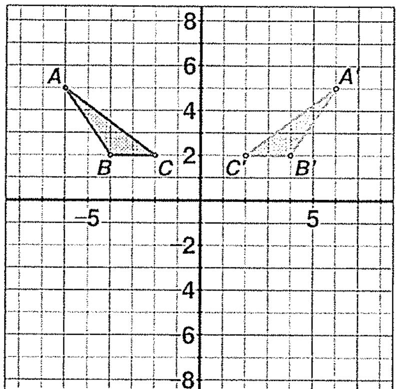
A Series of Transformations

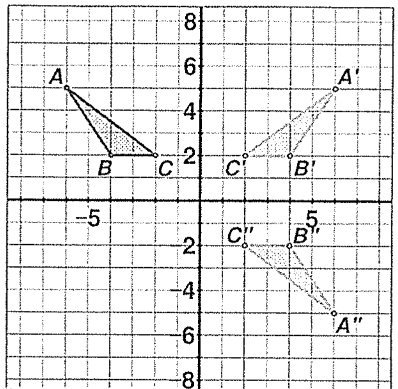
We can also apply multiple transformations to a shape. When we do so, we apply only one transformation at a time IN THE ORDER THEY ARE GIVEN. So if we wanted to dilate, translate, and then reflect we would first dilate the pre-image, then translate that new picture, then reflect that translation. For now we'll stick with just the two transformations we have covered so far.

Reflect-Reflect

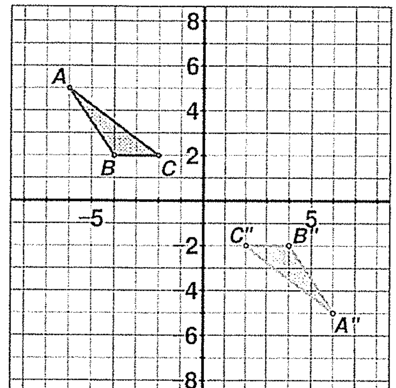
Taking a triangle, we can apply a reflection across the y-axis and then reflect that across the x-axis:



Reflection across y-axis



Reflect that across x-axis



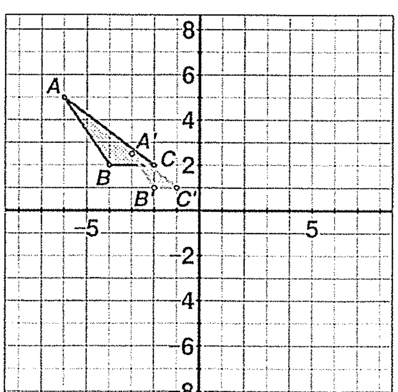
Final Picture of the Series

Notice that the in-between step is not shown on the final graph. You are welcome to leave the in-between step as you are graphing. Just be sure to label appropriately so we know which is which.

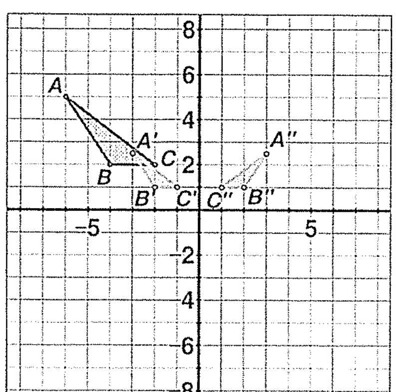
Also notice that we now have double prime to represent the second reflection and we end up with the final image points of A'': $(6,-5)$, B'': $(4,-2)$, C'': $(2,-2)$. You might notice that this also looks like a rotation by 180° which is true. It turns out that reflecting across both axes gives you a 180° turn.

Dilate-Reflect

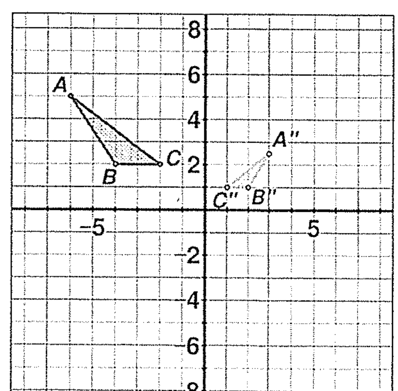
Let's again take that same triangle and dilate by $c = \frac{1}{2}$ with a center of dilation of (0,0) and then reflect across the y-axis. Keeping in mind the order, we must dilate first because that was the order listed. Then we'll take that dilated image and reflect it. Take a look. Shrink then fold and stamp that shrunken triangle.



Dilation by $c = \frac{1}{2}$



Reflect that across y-axis

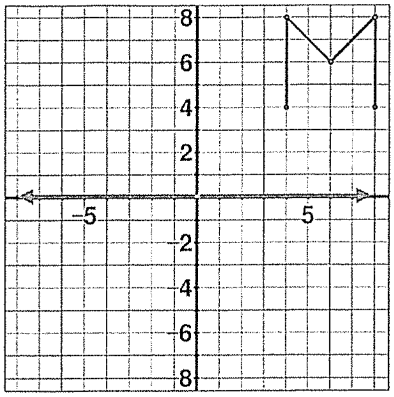


Final Picture of the Series

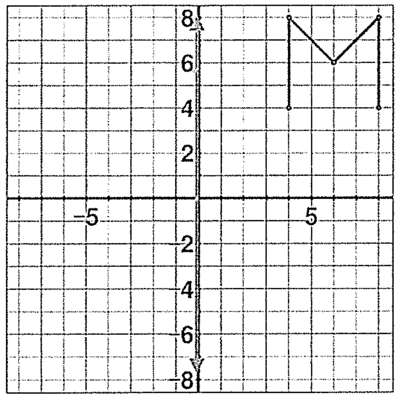
Lesson 2.2

Perform the given reflection or series of transformations on each given pre-image. The line of reflection is marked as a red line. When performing a dilation, use the origin as the center of dilation.

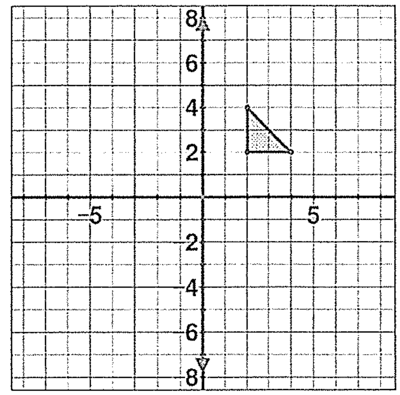
1. Reflect across x-axis



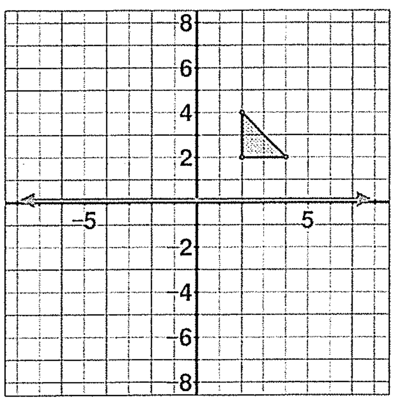
2. Reflect across y-axis



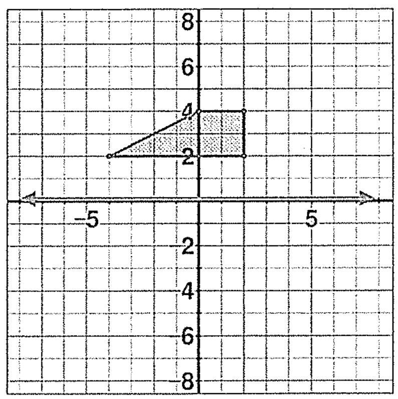
3. Reflect across y-axis



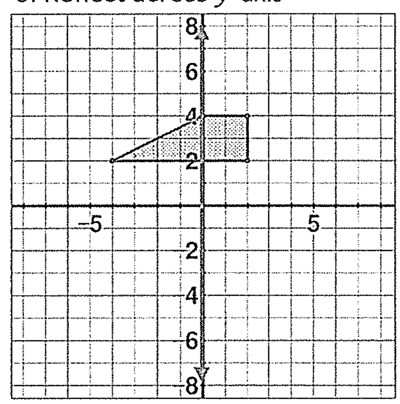
4. Reflect across x-axis



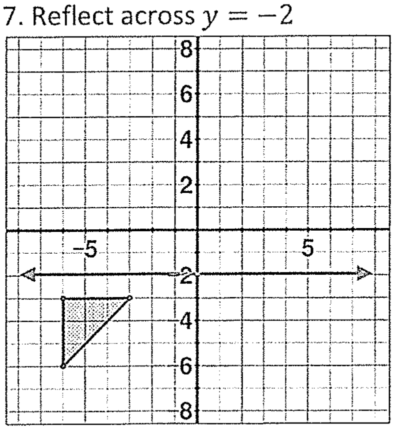
5. Reflect across x-axis



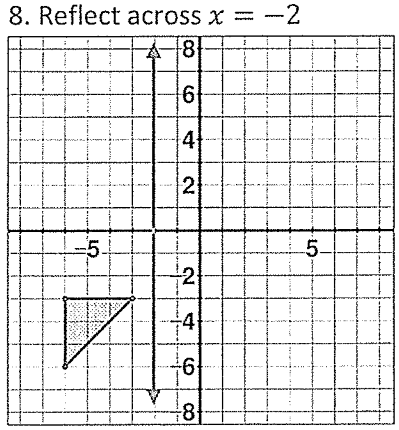
6. Reflect across y-axis



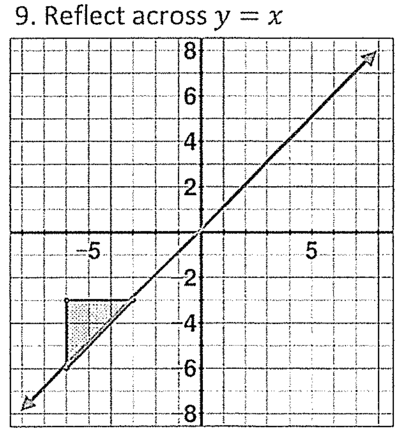
7. Reflect across $y = -2$



8. Reflect across $x = -2$



9. Reflect across $y = x$



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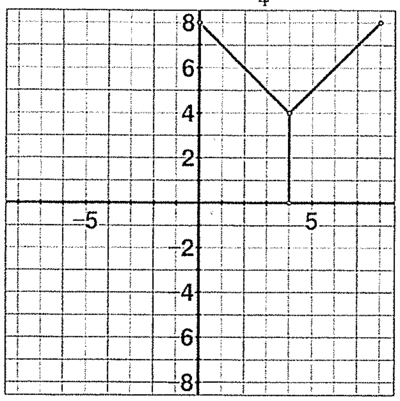
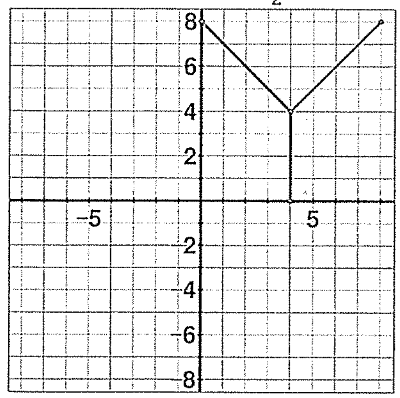
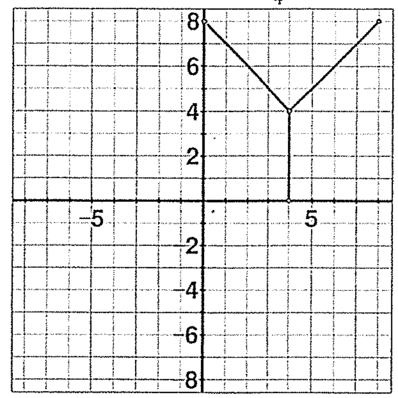
10. Reflect across x-axis and dilate by $c = \frac{1}{4}$



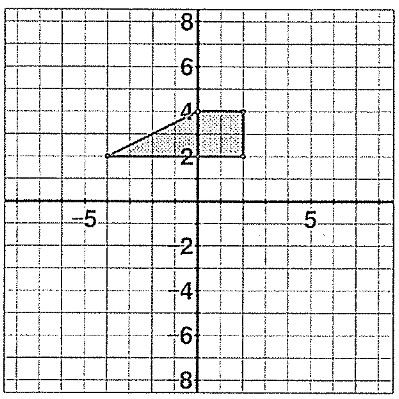
11. Reflect across y-axis and dilate by $c = \frac{1}{2}$



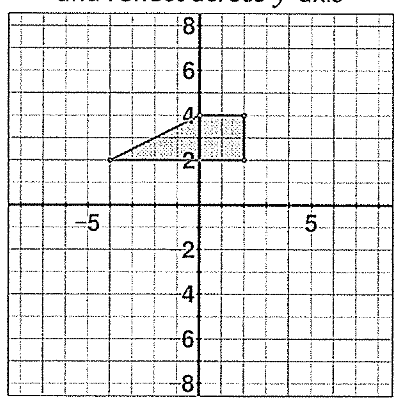
12. Reflect across y-axis and dilate by $c = \frac{3}{4}$



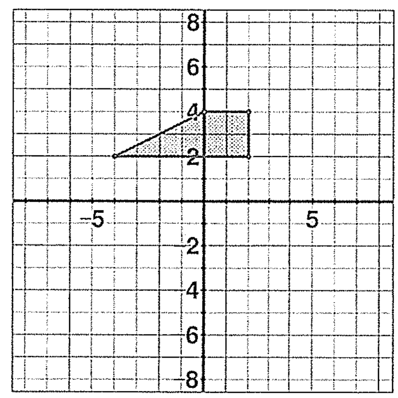
13. Dilate by $c = 2$ and reflect across x-axis



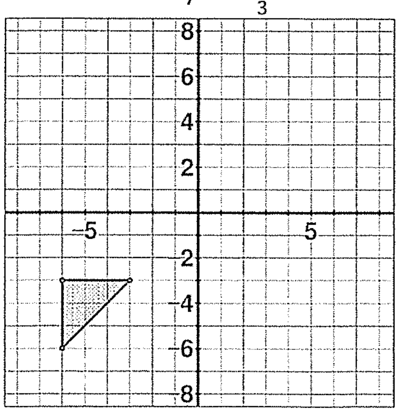
14. Dilate by $c = \frac{3}{2}$ and reflect across y-axis



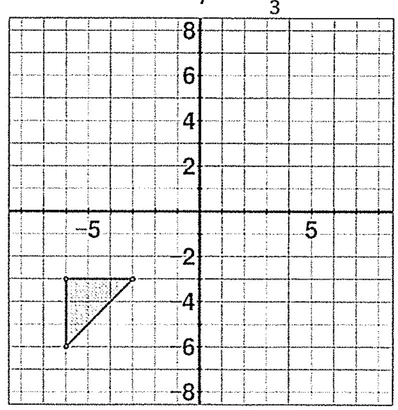
15. Dilate by $c = \frac{1}{2}$ and reflect across x-axis



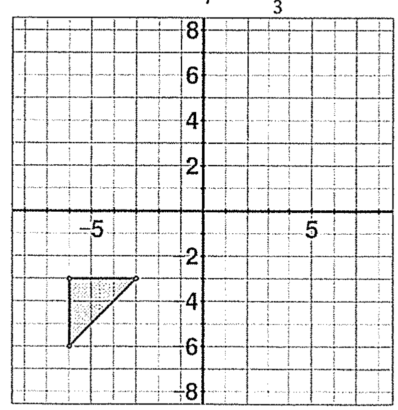
16. Reflect across x-axis and dilate by $c = \frac{1}{3}$



17. Reflect across y-axis and dilate by $c = \frac{2}{3}$



18. Reflect across x-axis and dilate by $c = \frac{4}{3}$



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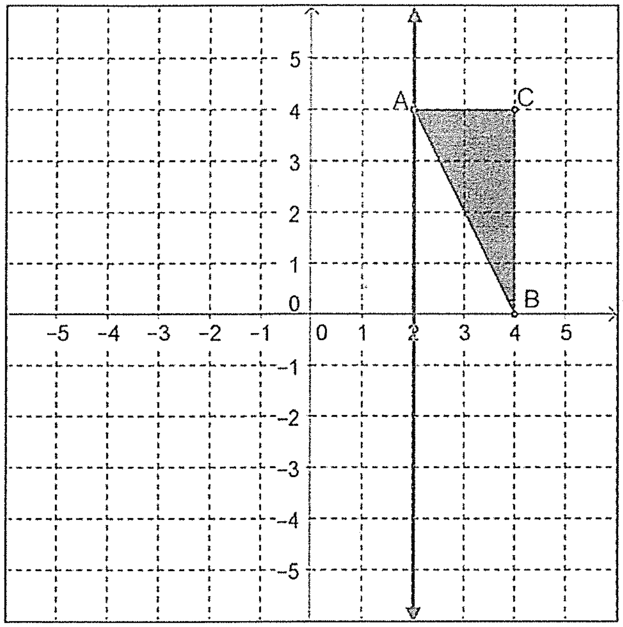


19. Reflect the triangle across the line $x = 2$. Which of the following statements will be true? For each explain why or why not.

a. Point A' is at the same coordinates as Point A.

b. Point B' is at the same coordinates as Point B.

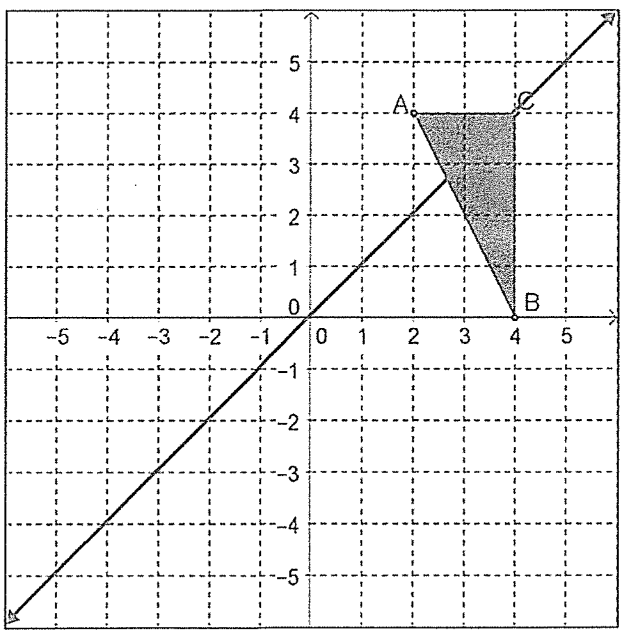
c. Point C' is at the same coordinates as Point C.

d. The image's perimeter equals the pre-image's.

e. The image's area equals the pre-image's.

f. The image is congruent to pre-image.

g. Line segment $\overline{A'C'}$ is horizontal.



20. Reflect the triangle across the line $y = x$. Which of the following statements will be true? For each explain why or why not.

a. Point A' will be at the same coordinates as Point A.

b. Point B' will be at the same coordinates as Point B.

c. Point C' will be at the same coordinates as Point C.

d. The image's perimeter equals the pre-image's.

e. The image's area equals the pre-image's.

f. The image is similar to pre-image.

g. Line segment $\overline{B'C'}$ is vertical.

21. In your own words, explain what a reflection does to a pre-image. Remember to consider the line of reflection in your explanation.

22. How could reflections be used in real life?

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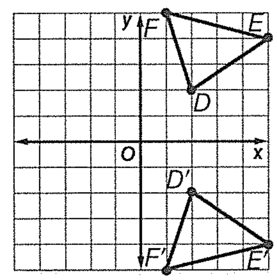
Reflections

A type of transformation where a figure is flipped over a line of reflection is a reflection. To reflect a figure over the *x*-axis, multiply the *y*-coordinates by −1. To reflect a figure over the *y*-axis, multiply the *x*-coordinates by −1.

Example

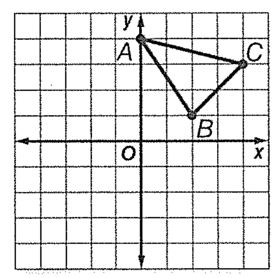
Triangle *DEF* has vertices *D*(2, 2), *E*(5, 4), and *F*(1, 5). Find the coordinates of the reflected image. Graph the figure and its reflected image over the *x*-axis.

Plot the vertices and connect to form △*DEF*. The *x*-axis is the line of symmetry. The distance from a point on △*DEF* to the line of symmetry is the same as the distance from the line of symmetry to the reflected image. The image coordinates are *D*(2, −2), *E*(5, −4), and *F*(1, −5).



Exercises

1. Triangle *ABC* has vertices *A*(0, 4), *B*(2, 1), and *C*(4, 3). Find the coordinates of the vertices of *ABC* after a reflection over the *x*-axis. Then graph the figure and its reflected image.

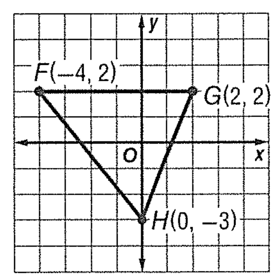


For Exercises 2 and 3, use the following information.

Triangle *FGH* has vertices *F*(−4, 2), *G*(2, 2), and *H*(0, −3).

2. What are the coordinates of the image of point *F* after a reflection over the *x*-axis?

3. Graph triangle *FGH* and its image after a reflection over the *x*-axis.



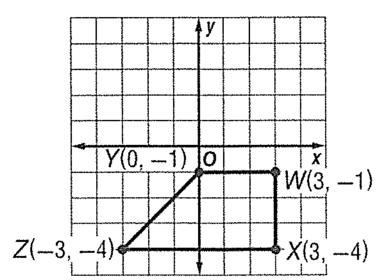
For Exercises 4–6, use the following information.

Quadrilateral *WXYZ* has vertices *W*(3, −1), *X*(3, −4), *Y*(0, −1), and *Z*(−3, −4).

4. What are the coordinates of the image of point *W* after a reflection over the *y*-axis?

5. What are the coordinates of the image of point *X* after a reflection over the *y*-axis?

6. Graph quadrilateral *WXYZ* and its image after a reflection over the *y*-axis.



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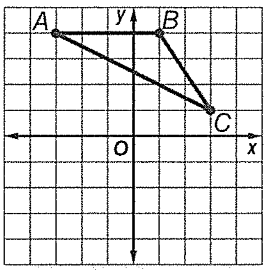
NAME ______________________ DATE ____________ PERIOD ________

Skills Practice

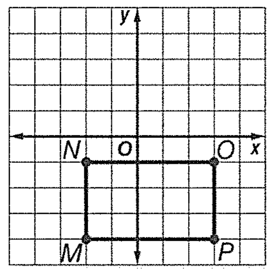
Reflections

Graph the figure and its reflection over the x-axis. Then find the coordinates of the reflected image.

1. triangle ABC with vertices $A(-3, 4)$, $B(1, 4)$, and $C(3, 1)$

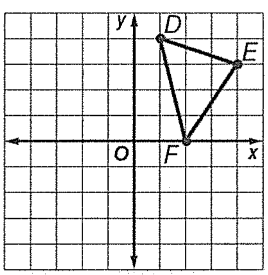


2. rectangle $MNOP$ with vertices $M(-2, -4)$, $N(-2, -1)$, $O(3, -1)$, and $P(3, -4)$

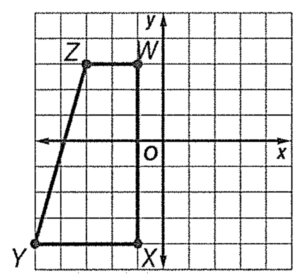


Graph the figure and its reflection over the y-axis. Then find the coordinates of the reflected image.

3. triangle DEF with vertices $D(1, 4)$, $E(4, 3)$, and $F(2, 0)$

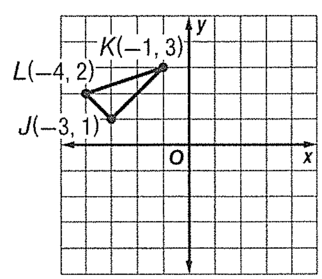


4. trapezoid $WXYZ$ with vertices $W(-1, 3)$, $X(-1, -4)$, $Y(-5, -4)$, and $Z(-3, 3)$



For Exercises 5–8, use the following information.

Triangle JKL has vertices $J(-3, 1)$, $K(-1, 3)$, and $L(-4, 2)$.



5. What are the coordinates of the image of point J after a reflection over the y-axis?

6. What are the coordinates of the image of point K after a reflection over the y-axis?

7. What are the coordinates of the image of point L after a reflection over the y-axis?

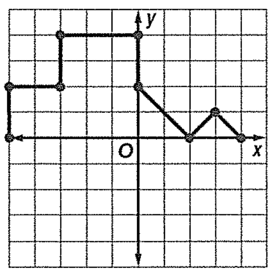
8. Graph triangle JKL and its image after a reflection over the y-axis.

NAME ______________________ DATE ____________ PERIOD ________

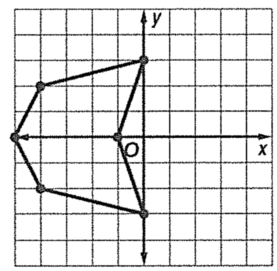
Problem-Solving Practice

Reflections

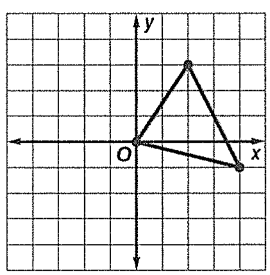
1. **DESIGNS** Half of a design is shown below. Reflect the figure across the x-axis to obtain the completed design.



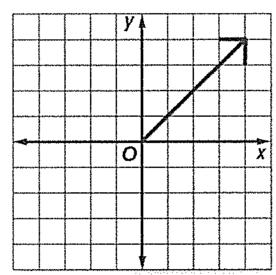
2. **DESIGNS** Half of a design is shown below. Reflect the figure across the y-axis to obtain the completed design.



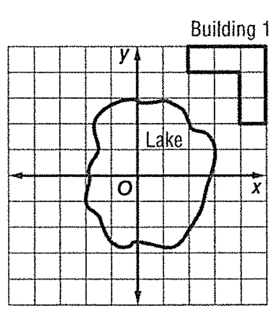
3. **LOGO** Half of a logo is shown below. Reflect the figure across the y-axis to obtain the completed figure.



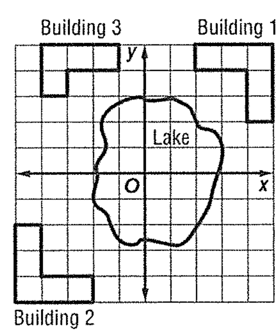
4. **SYMBOLS** The figure shows a ray plotted on a coordinate system. Reflect the ray across the x-axis. Graph the reflected image.



5. **ARCHITECTURE** A corporate plaza is to be built around a small lake. Building 1 has already been built. Suppose there are axes through the lake as shown. Show where Building 2 should be built if it will be a reflection of Building 1 across the y-axis followed by a reflection across the x-axis.



6. **ARCHITECTURE** Use the information from Exercise 5. Suppose that a third building is to be built as shown. To complete the business park, show where a fourth building should be built if it is a reflection of Building 3 across the x- and y-axis.



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2.3 Constructing Rotations

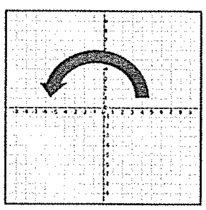
We're halfway through the transformations and our next one, the rotation, gives a congruent image just like the reflection did. Just remember that a series of transformations with a rotation or reflection doesn't guarantee that it will be congruent because it could have a dilation in there. That would make the image similar to the pre-image instead of congruent.

Rotations

At the 8th grade, we will usually be rotating about the origin and only in increments of 90°. That means that we could rotate 90°, 180° or 270°. We won't be doing a 360° because that would turn the shape in a complete circle giving us an image right on top of the pre-image. That's rather boring, so we'll stick with the other three. The point around which we rotate is called the **center of rotation**.

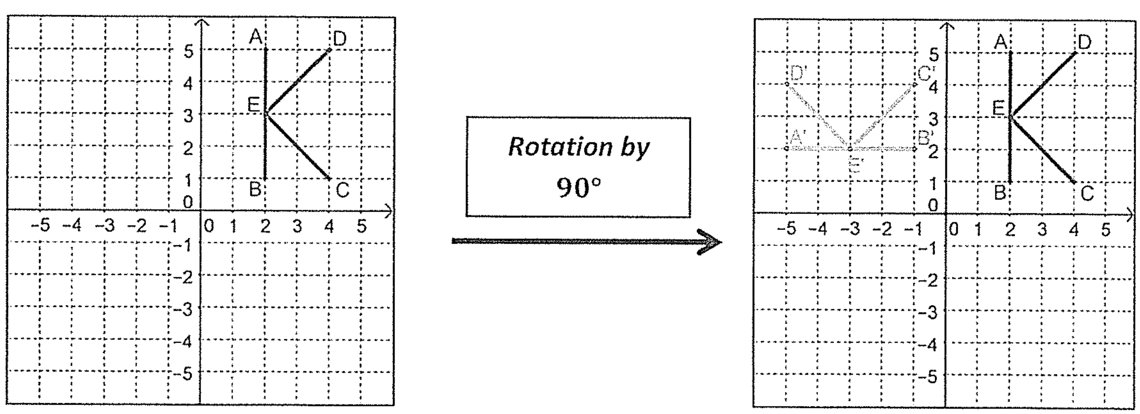
Rotations are also known as turns. So we may say a ninety degree turn, or a quarter turn, at times. Another name for a rotation would be a spin because we are spinning the picture about the origin.

90° Rotation about the Origin

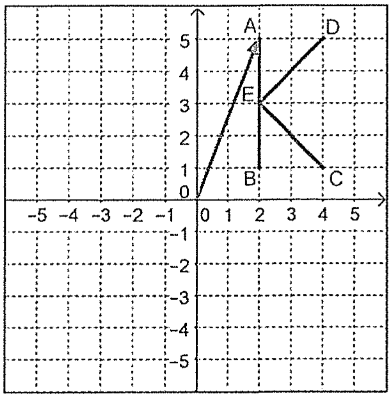


In mathematics we always rotate counterclockwise unless you are specifically told otherwise. That means we rotate like the arrow to the left.

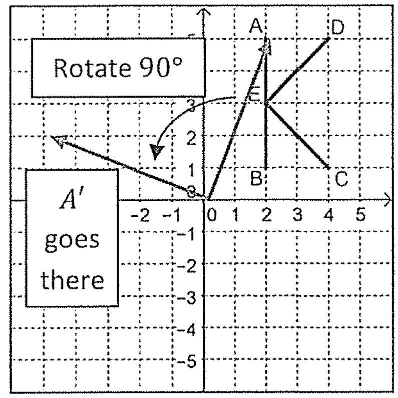
Looking at a capital letter may be an easy example to think about to make sure we get the concept of rotations down. Take the following capital "K" and rotate it counterclockwise 90° in your mind using the origin as the center of rotation. You can also do so physically by putting your finger on the origin and turning the book a quarter turn counterclockwise. You should be able to visualize in your mind the "back" of the K turning to lie flat parallel to the x-axis. Here's what it would look like:

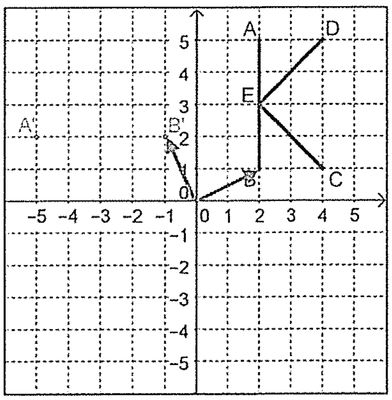


How did that happen? We can use the formulas to execute a 90° turn, but again, we don't want to just be memorize tons of formulas. Instead, imagine a clock hand coming out from the center of rotation (the origin) and pointing towards each point. We'll start with the clocking hand pointing at Point A.

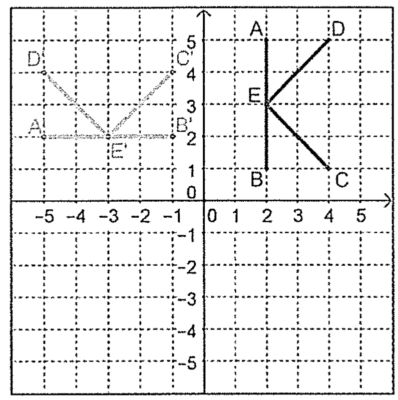


Notice the clock hand point towards A in the picture to the left. Now imagine rotating that clock hand 90° counterclockwise (or to the left). It should be pointing like we see in the picture to the right. The clock hand is now pointing at the place on the coordinate plane where we should put the image point A'. We can keep doing this a point at a time until we get the whole image built.



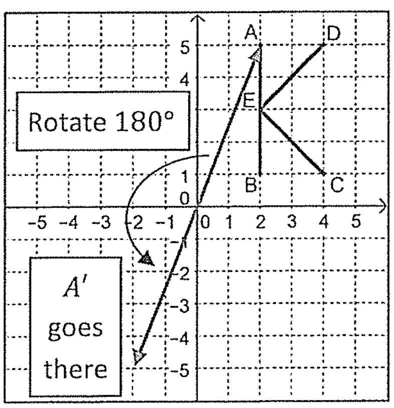


Let's do the same thing with point B. Draw your clock hand and rotate it 90° counterclockwise as shown to the left. After getting where B' is at, you can probably construct the rest of the letter "K" picture because you now know where the back of the letter is. The final picture should look like the one to the right.



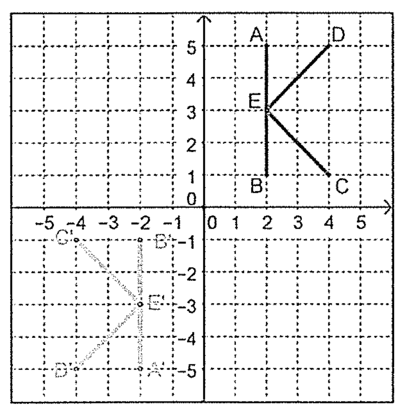
180° or 270° Rotation about the Origin

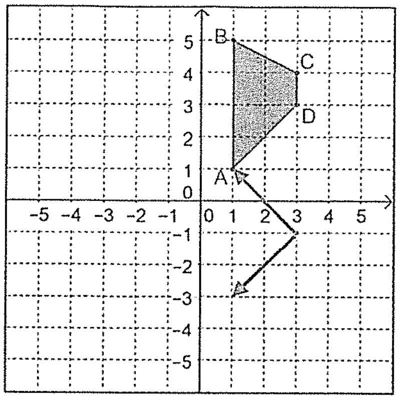
We can use the same method for making other rotations. Just remember that 180° is half a circle and that 270° a three-quarters turn. Perhaps the easiest way to think about a 270° rotation is just a backwards 90° rotation.



Let's start by rotating the "K" 180° and think of a clock hand going to point A again. Rotate that clock hand 180° and it will point to where A' should be. Going point by point we get our final image.

The same method would work for a 270° rotation.

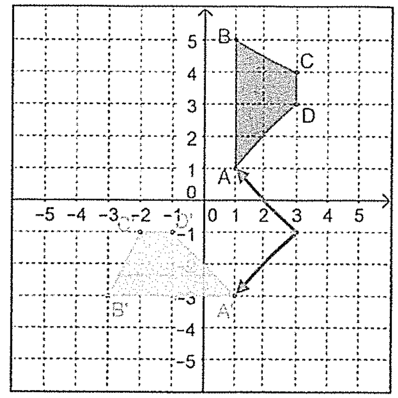




Using Other Centers of Rotation

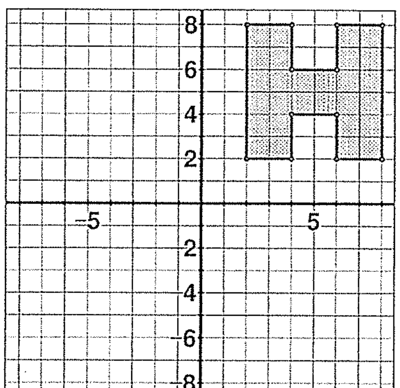
By thinking about rotations as turning clock hands for each point, we can actually rotate around any center of rotation. Let's take a trapezoid and rotate it 90° about a center of rotation of $(3, -1)$.

Start by putting a clock hand out to point A and rotating just that clock hand the 90° find where point A' will be. Continuing point by point we get the final shape to the right.

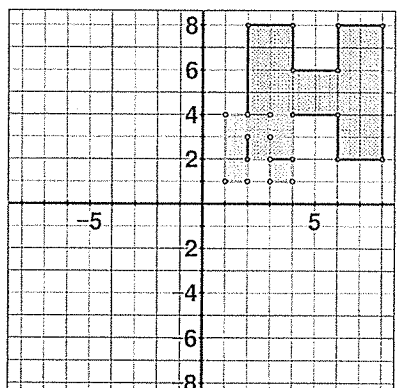


A Series Transformations

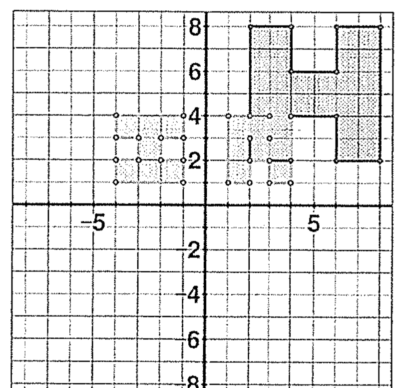
Let's take the following block letter "H" and dilate it by $c = \frac{1}{2}$ using the origin as the center of dilation, rotate it 90° using the origin as the center of rotation, and then reflect it across the x-axis. That's three transformations in a row, which is perfectly acceptable. Here's what it would look like a step at a time:



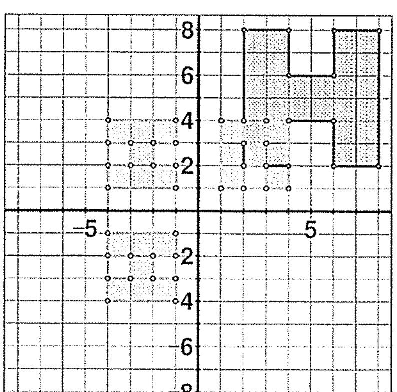
Start with the "H"



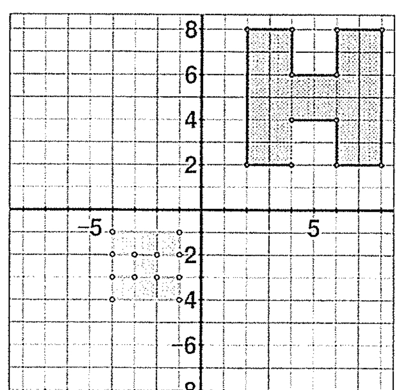
Dilate by $c = \frac{1}{2}$



Rotate by 90°



Reflect across x-axis



Final Image of Series

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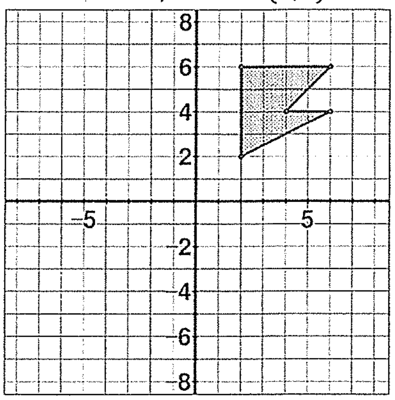
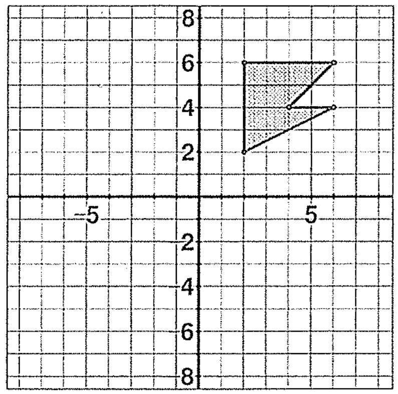
Lesson 2.3

Perform the given rotation or series of transformations on each given pre-image. When performing a dilation or rotation, use the origin as the center of dilation or rotation unless it is specified otherwise.

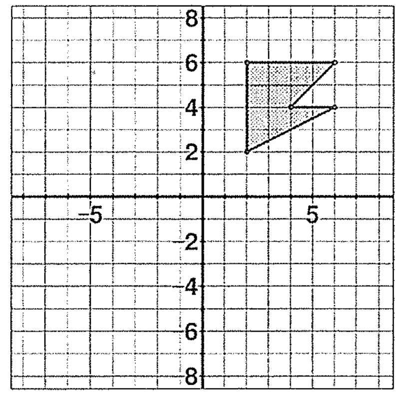
1. Rotate 90°, center: (0,0)



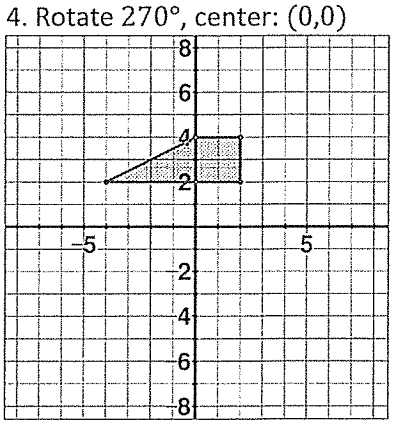
2. Rotate 180°, center: (0,0)



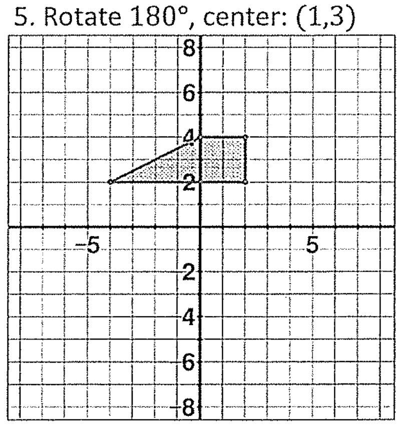
3. Rotate 270°, center: (0,0)



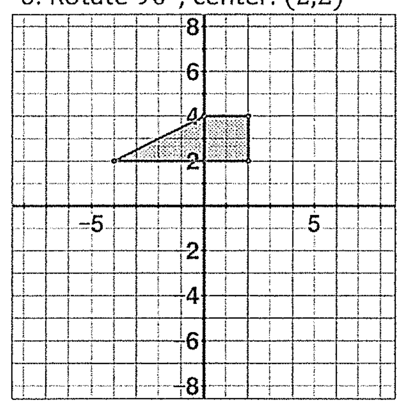
4. Rotate 270°, center: (0,0)



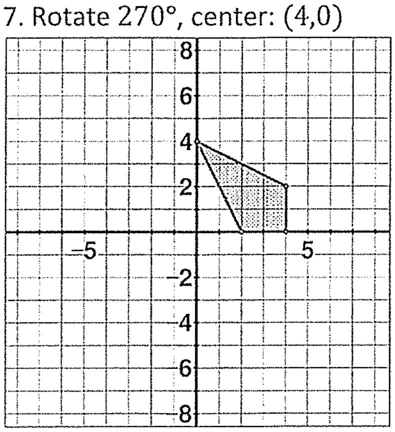
5. Rotate 180°, center: (1,3)



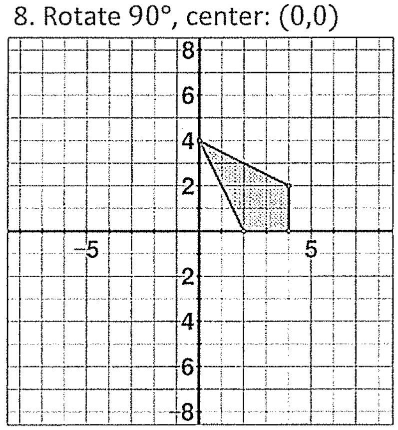
6. Rotate 90°, center: (2,2)



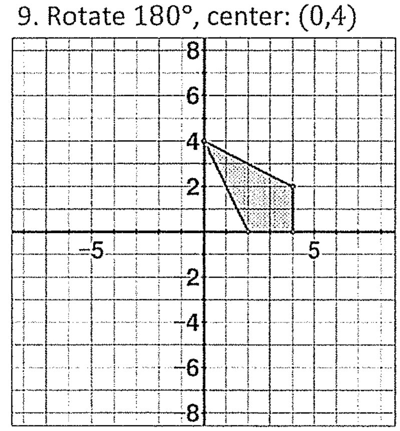
7. Rotate 270°, center: (4,0)



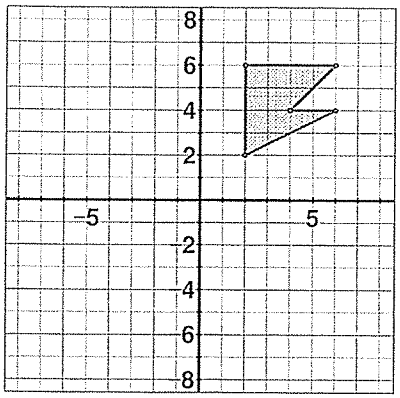
8. Rotate 90°, center: (0,0)



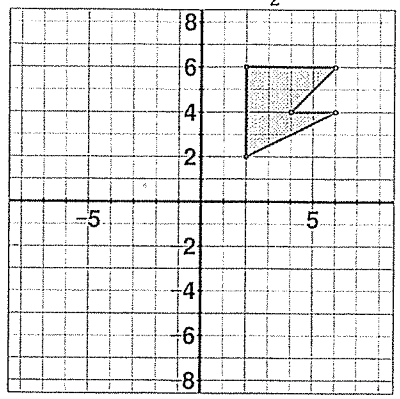
9. Rotate 180°, center: (0,4)



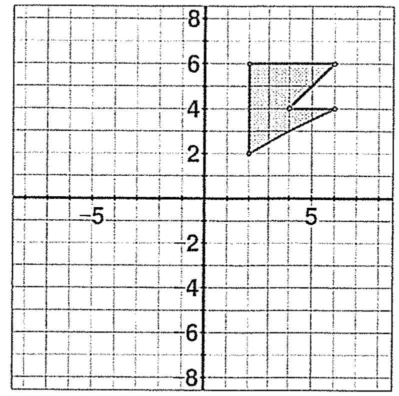
10. Rotate 90°
and reflect across x-axis



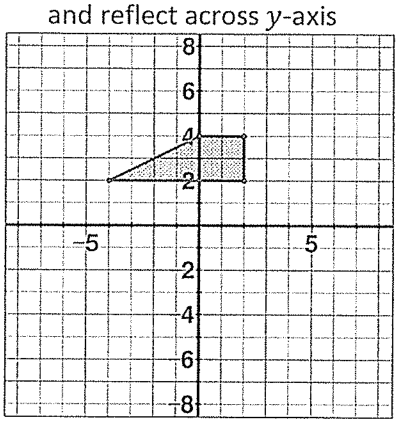
11. Rotate 180°
and dilate by $c = \frac{1}{2}$



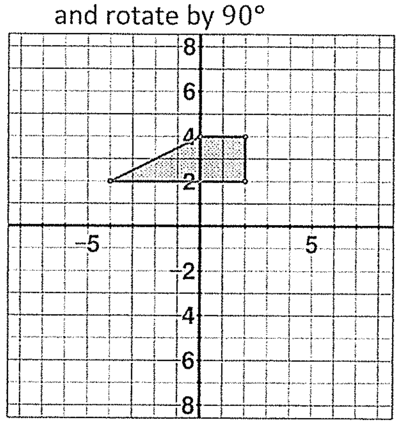
12. Rotate 270°
and reflect across y-axis



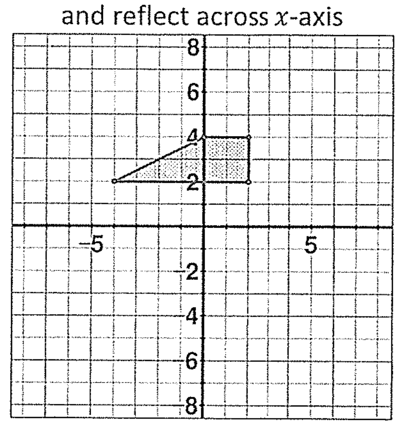
13. Dilate by $c = 2$
and reflect across y-axis



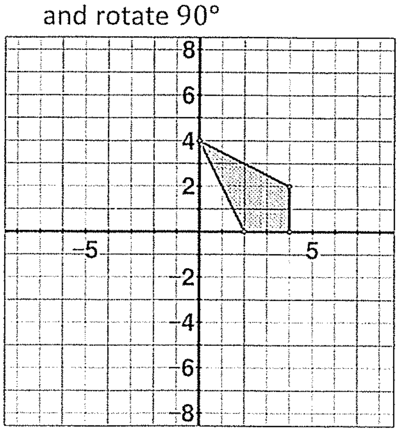
14. Dilate by $c = \frac{1}{2}$
and rotate by 90°



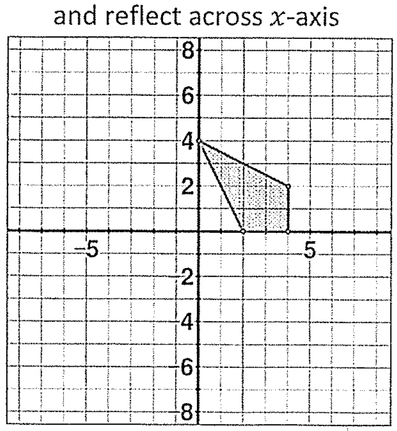
15. Rotate 180°
and reflect across x-axis



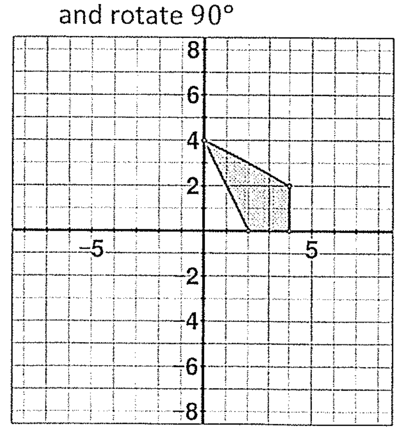
16. Reflect across y-axis
and rotate 90°



17. Dilate by $c = \frac{1}{2}$
and reflect across x-axis



18. Dilate by $c = 2$
and rotate 90°



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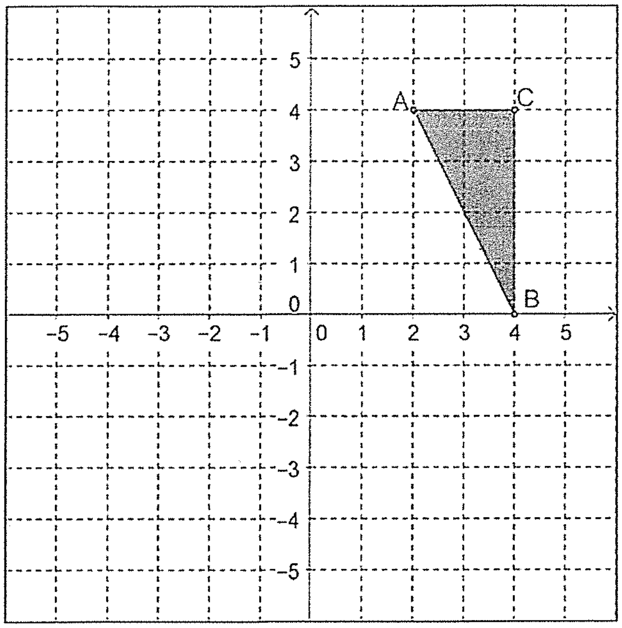


19. Rotate the triangle 90° around the center of rotation (3,2). Which of the following statements will be true? For each explain why or why not.

a. Point A' is at the same coordinates as Point A.

b. Point B' is at the same coordinates as Point B.

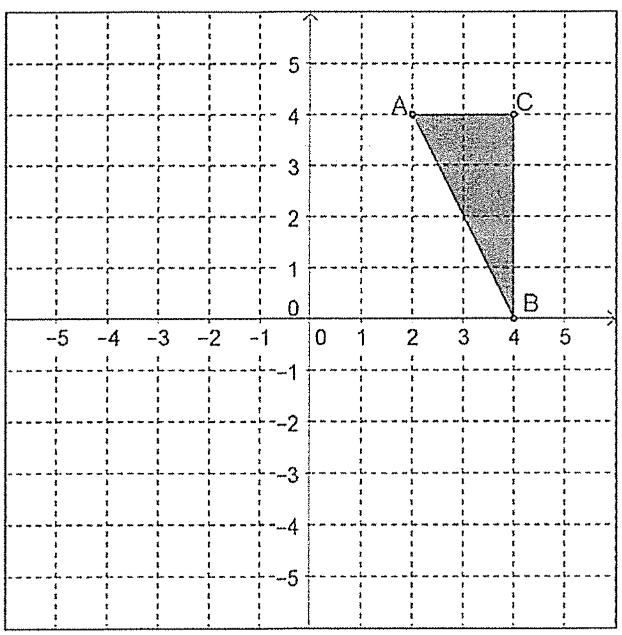
c. Point C' is at the same coordinates as Point C.

d. The image's perimeter equals the pre-image's.

e. The image's area equals the pre-image's.

f. The image is congruent to pre-image.

g. Line segment $\overline{A'C'}$ is horizontal.



20. Rotate the triangle 180° around the center of rotation (4,0). Which of the following statements will be true? For each explain why or why not.

a. Point A' will be at the same coordinates as Point A.

b. Point B' will be at the same coordinates as Point B.

c. Point C' will be at the same coordinates as Point C.

d. The image's perimeter equals the pre-image's.

e. The image's area equals the pre-image's.

f. The image is similar to pre-image.

g. Line segment $\overline{B'C'}$ is vertical.

21. In your own words, explain what a rotation does to a pre-image. Remember to consider the center of rotation in your explanation.

22. How could rotations be used in real life?

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NAME ________________________________ DATE ______________ PERIOD ________

Reteach

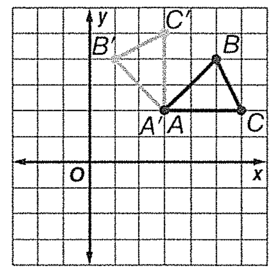
Rotations

A rotation is a transformation in which a figure is rotated, or turned, about a fixed point. The center of rotation is the fixed point. The preimage and the image are congruent.

Example 1

Triangle ABC has vertices $A(3, 2)$, $B(5, 4)$, $C(6, 2)$. Graph the figure and its image after a counterclockwise rotation of 90° about vertex A. Then give the coordinates of the vertices for $A'B'C'$.

Step 1 Graph the original triangle.



Step 2 Graph the rotated image. Use a protractor to measure an angle of 90° with B as one point on the ray and A as the vertex. Mark off a point the same distance as $\overline{BA}$. Label this point B' as shown.

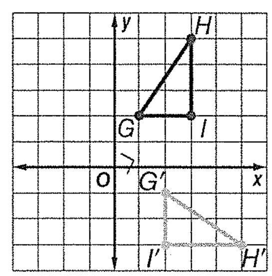
Step 3 Repeat Step 2 for point C. Since A is the point at which $\triangle ABC$ is rotated, A' will be in the same position as A.

So, the coordinates of the vertices of $\triangle ABC$ are $A'(3, 2)$, $B'(1, 4)$, $C'(3, 5)$.

Example 2

Triangle GHI has vertices $G(1, 2)$, $H(3, 5)$, $I(3, 2)$. Graph the figure and its image after a clockwise rotation of 90° about the origin. Then give the coordinates of the vertices for $\triangle G'H'I'$.

Step 1 Graph $\triangle GHI$ on a coordinate plane.



Step 2 Sketch $\overline{GO}$ connecting point G to the origin. Sketch another segment, $G'O'$ so that the angle between point G, O, and G' measures 90° and the segment is congruent to $\overline{GO}$.

Step 3 Repeat Step 2 for points H and I. Then connect the vertices to form $\triangle G'H'I'$.

So, the coordinates of the vertices of $\triangle G'H'I'$ are $G'(2, -1)$, $H'(5, -3)$, and $I'(2, -3)$.

Exercises

Refer to $\triangle GHI$ in Example 2 above. Graph GHI after each rotation. Then give the coordinates of the vertices for $G'H'I'$.

1. 180° counterclockwise about vertex G

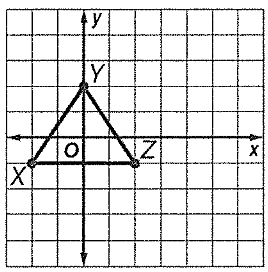
2. 180° clockwise about the origin

Skills Practice

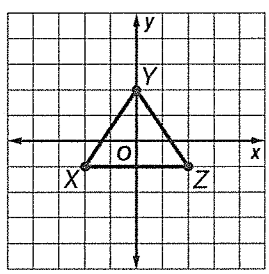
Rotations

For Exercises 1 and 2, graph $\triangle XYZ$ and its image after each rotation. Then give the coordinates of the vertices for $\triangle X'Y'Z'$.

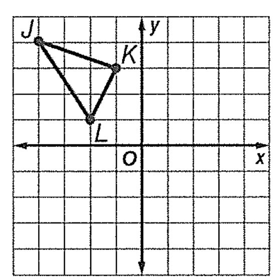
1. 180° clockwise about vertex Z



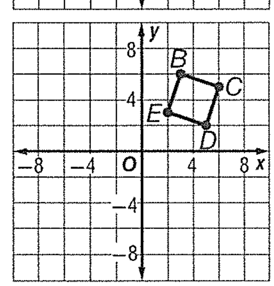
2. 90° clockwise about vertex X



3. Triangle JKL has vertices $J(-4, 4)$, $K(-1, 3)$, and $L(-2, 1)$. Graph the figure and its rotated image after a clockwise rotation of 90° about the origin. Then give the coordinates of the vertices for triangle $J'K'L'$.



4. Quadrilateral $BCDE$ has vertices $B(3, 6)$, $C(6, 5)$, $D(5, 2)$, and $E(2, 3)$. Graph the figure and its rotated image after a counterclockwise rotation of 180° about the origin. Then give the coordinates of the vertices for quadrilateral $B'C'D'E'$.



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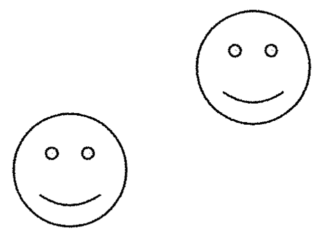
Problem-Solving Practice

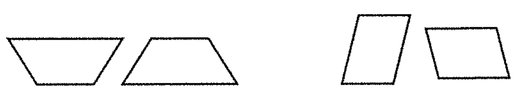
Rotations

1. **OPEN-ENDED** Draw a figure that has rotational symmetry with 90° and 180° as its angles of rotation.

2. **CLASSIFY** Identify the transformation shown below as a translation, reflection, or rotation. Explain.



3. **ROTATIONS** Which figure below was rotated 90° counterclockwise?



4. **LETTERS** Which capital letters in the word TRANSFORMATION produce the same letter after being rotated 180°?

5. **REAL-WORLD** Describe a real-world example of where you could find a rotation.

6. **ART** An art design is shown. State the angles of rotation.

2.4 Constructing Translations

Our final transformation also creates images that are congruent to their pre-images. Translations are also sometimes called a "slide" because it appears that they slide the pre-image to a new position on the coordinate plane.

Translation Vectors

To translate a shape, we use a **translation vector**. The translation vector looks like this: $\binom{a}{b}$ where a represents how much to add to the x-value of each point in the pre-image and b represents how much to add to the y-value of each point in the pre-image. This leads to the formulas:

$$x' = x + a$$

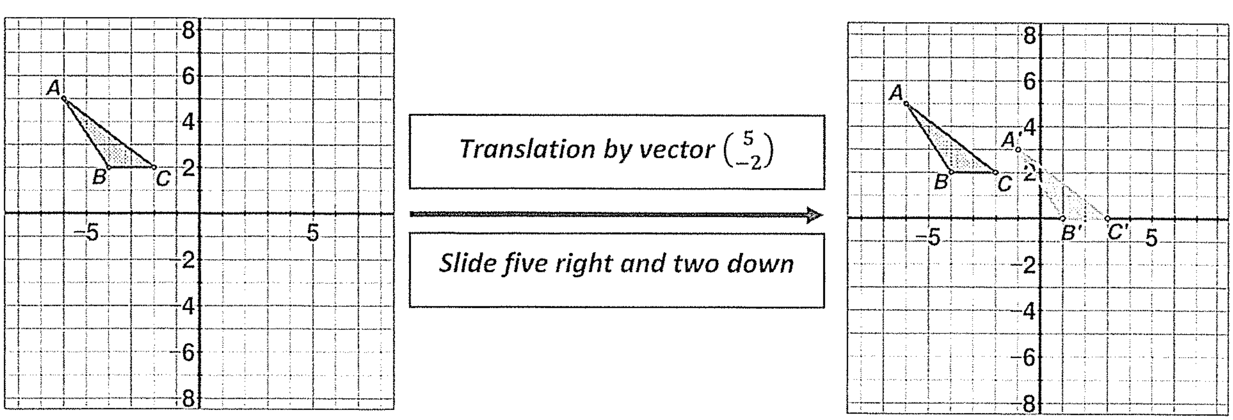
$$y' = y + b$$

Let's look at a couple of specific vectors to get the main idea down. The vector $\binom{2}{-3}$ would slide a shape two to the right and three down. Remember that the top number in the vector is added to the x-coordinate. That means left or right movement. Since it's positive two, the shape will move two to the right. The bottom number of the vector is added to the y-coordinate and therefore represents up and down movement. Since it's a negative three, the shape will move three down.

In the same manner, the vector $\binom{-1}{0}$ would slide a shape one left and not at all up or down. This is just a horizontal slide. A vertical slide would have a zero in the top of the vector. Let's look at some examples with specific pre-images.

Let the pre-image be the points $A\colon(-6,5), B\colon(-4,2), C\colon(-2,2)$ giving us a triangle and let's translate that using the translation vector $\binom{5}{-2}$. This means add five to each x-coordinate and negative two to each y-coordinate or move the triangle five right and two down.

To get our new points, we do the addition for each point in the pre-image. $A'\colon(-6+5, 5+(-2))$ or $A'\colon(-1,3)$. Similarly we find that $B'\colon(1,0)$ and $C'\colon(3,0)$. Instead of using the formulas, we can always just count five right and two down for each pre-image point and plot the new image point. The final result should be as follows:



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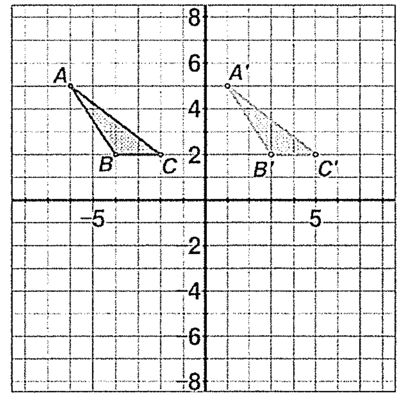
A Horizontal Slide

Sometimes throwing in a zero can confuse people, so let's work through a horizontal translation with the vector $\binom{7}{0}$. Notice that this will slide the pre-image shape seven to the right. Let's use the same triangle as before. You should see that the new points will be A': (1,5), B': (3,2), C': (5,2) as seen to the right:



A Series of Transformations

Just as with all previous examples, we can add a translation to a series of transformations to make more interesting pictures. Keep in mind that if the series contains only reflections, rotations and/or translations, the image will be congruent to the pre-image. If we use a dilation at any point, we will end up with a similar, not congruent, image.

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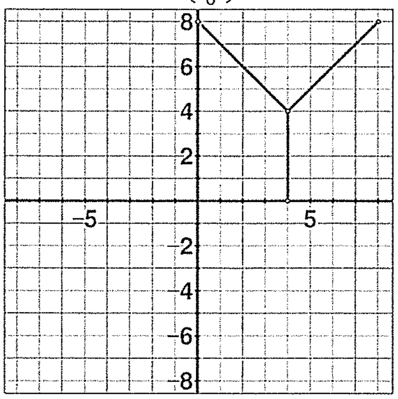
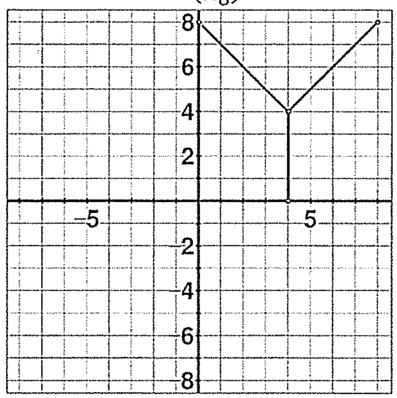
Lesson 2.4

Perform the given translation or series of transformations on each given pre-image. When performing a dilation or rotation, use the origin as the center of dilation or rotation.

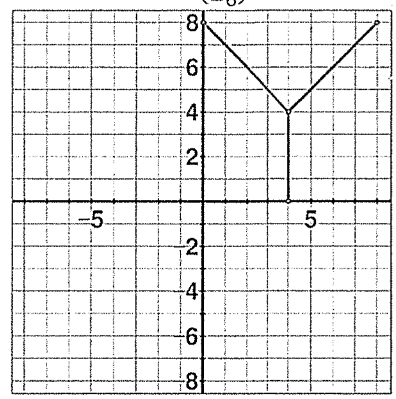
1. Translate by $\begin{pmatrix} -8 \\ 0 \end{pmatrix}$



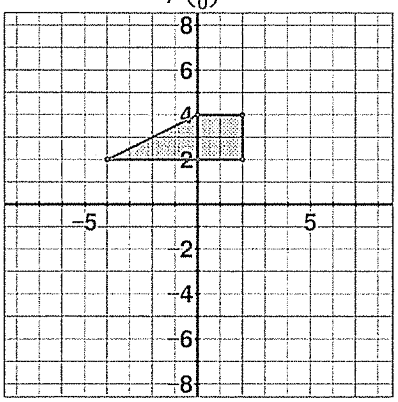
2. Translate by $\begin{pmatrix} 0 \\ -8 \end{pmatrix}$



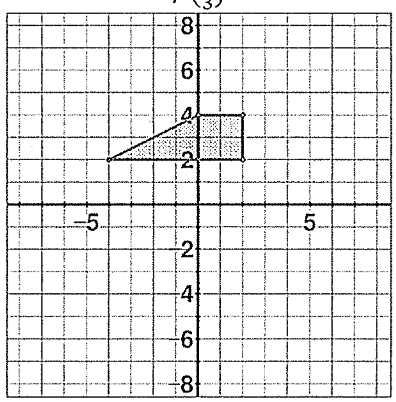
3. Translate by $\begin{pmatrix} -3 \\ -6 \end{pmatrix}$



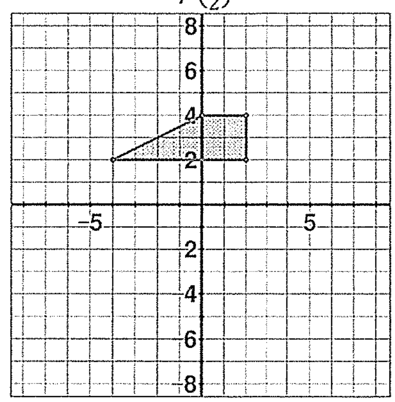
4. Translate by $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$



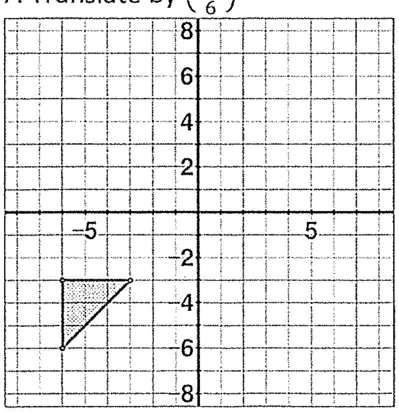
5. Translate by $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$



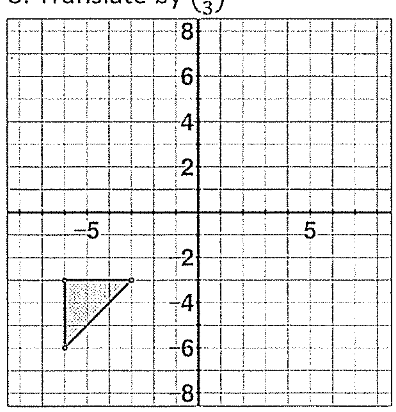
6. Translate by $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$



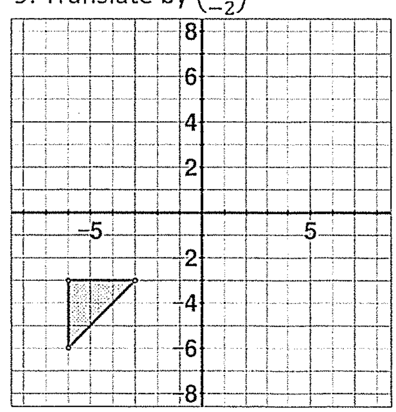
7. Translate by $\begin{pmatrix} -2 \\ 6 \end{pmatrix}$



8. Translate by $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$



9. Translate by $\begin{pmatrix} 8 \\ -2 \end{pmatrix}$



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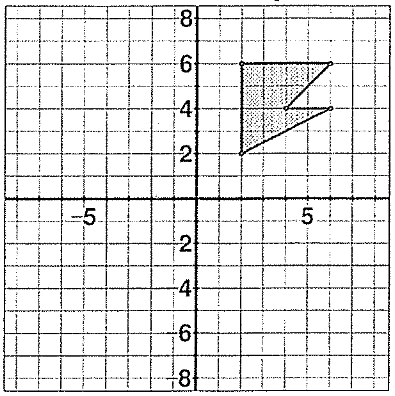
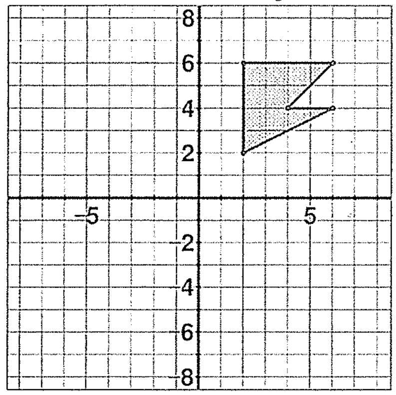
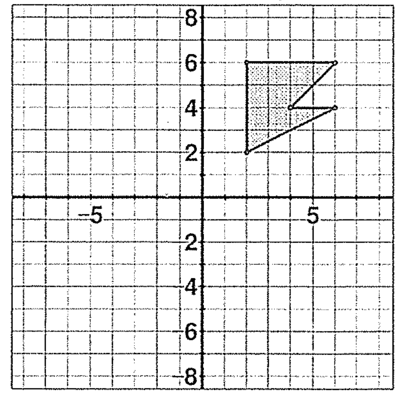
10. Rotate 90° and translate by $\binom{3}{-4}$



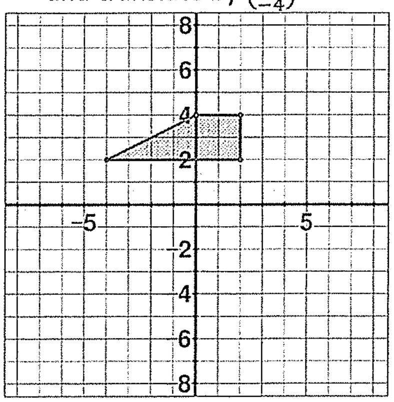
11. Rotate 180° and translate by $\binom{-1}{8}$



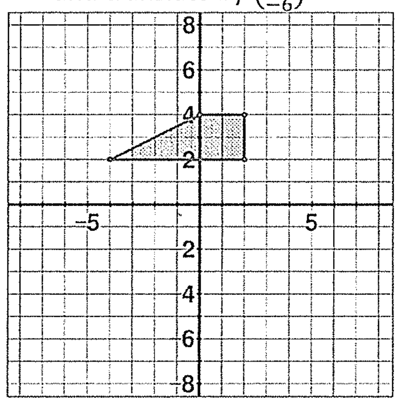
12. Reflect across x-axis and translate by $\binom{-8}{8}$



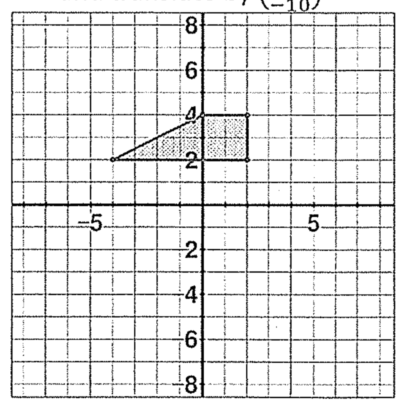
13. Reflect across y-axis and translate by $\binom{0}{-4}$



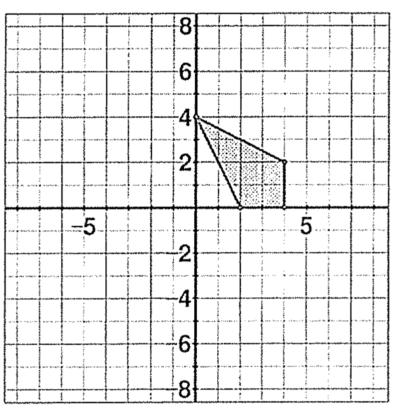
14. Dilate by $c = \frac{1}{2}$ and translate by $\binom{4}{-6}$



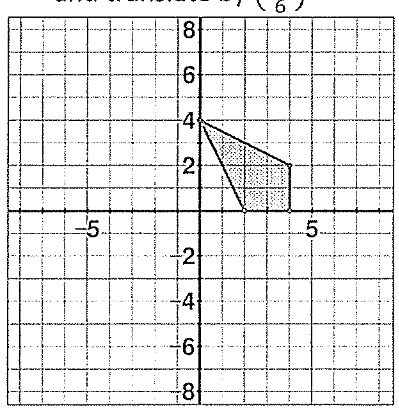
15. Dilate by $c = 2$ and translate by $\binom{2}{-10}$



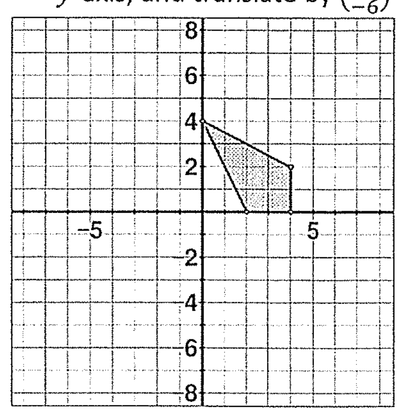
16. Dilate by $c = \frac{1}{2}$, rotate 90°, and reflect across x-axis



17. Dilate by $c = \frac{1}{2}$, rotate 180°, and translate by $\binom{-4}{6}$



18. Dilate by $c = 2$, reflect across y-axis, and translate by $\binom{4}{-6}$



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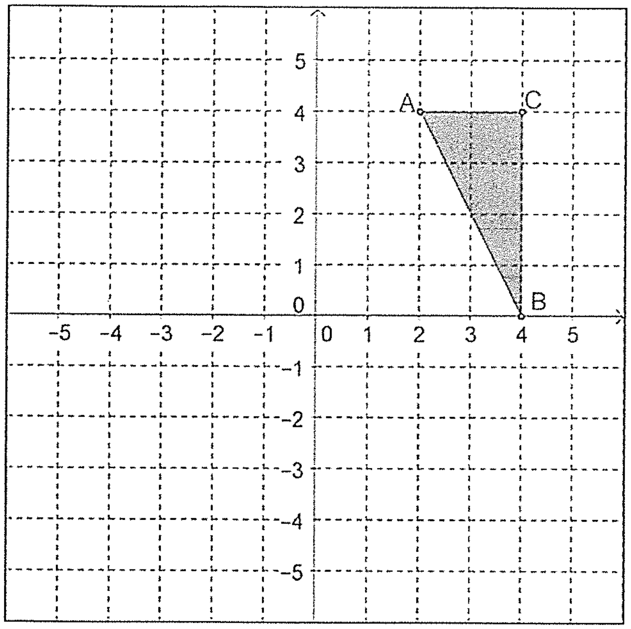


19. Translate the triangle by vector $\binom{-4}{-3}$. Which of the following statements will be true? For each explain why or why not.

a. Point A' is at the same coordinates as Point A.

b. Point B' is at the same coordinates as Point B.

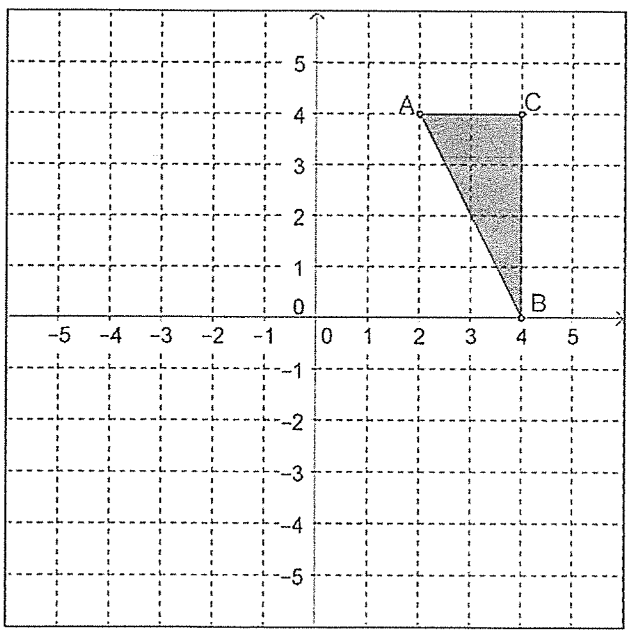
c. Point C' is at the same coordinates as Point C.

d. The image's perimeter equals the pre-image's.

e. The image's area equals the pre-image's.

f. The image is congruent to pre-image.

g. Line segment $\overline{A'C'}$ is horizontal.



20. Translate the triangle by vector $\binom{0}{1}$. Which of the following statements will be true? For each explain why or why not.

a. Point A' will be at the same coordinates as Point A.

b. Point B' will be at the same coordinates as Point B.

c. Point C' will be at the same coordinates as Point C.

d. The image's perimeter equals the pre-image's.

e. The image's area equals the pre-image's.

f. The image is similar to pre-image.

g. Line segment $\overline{B'C'}$ is vertical.

21. In your own words, explain what a translation does to a pre-image. Remember to consider the translation vector in your explanation.

22. How could translations be used in real life?

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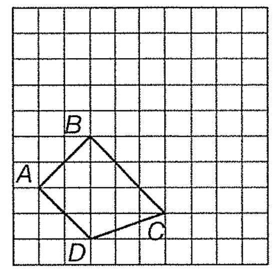
Reteach

Translations

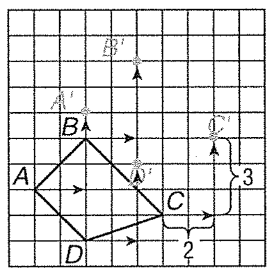
When a figure is translated, each point is moved the same distance and in the same direction. The translated figure is congruent to the original figure and has the same orientation.

Example

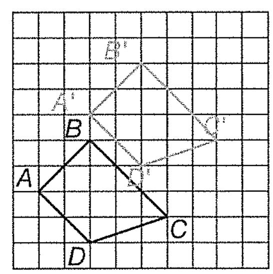
Draw the image of quadrilateral *ABCD* after a translation 2 units right and 3 units up.



Step 1 To find the corresponding point for vertex *A*, start at *A* and move 2 units to the right along the horizontal grid line and then move up 3 units along the vertical grid line. Draw a point and label it *A′*. Repeat for each vertex.



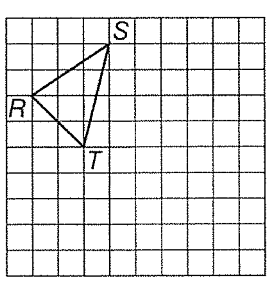
Step 2 Connect the new vertices to form quadrilateral *A′B′C′D′*.



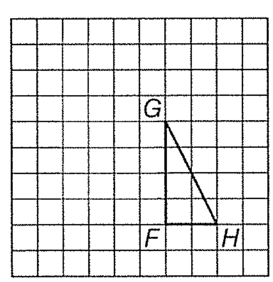
Exercises

Draw the image of the figure after the indicated translation.

1. 5 units right and 4 units down



2. 3 units left and 2 units up



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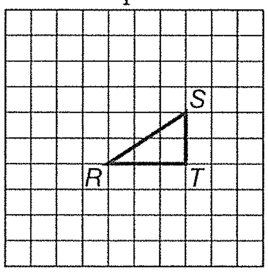
Skills Practice

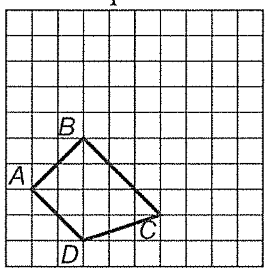
Translations

Graph the image of the figure after the indicated translation.

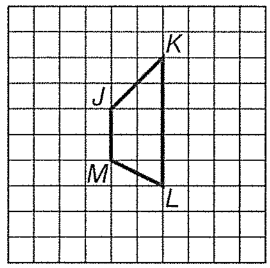
1. 2 units left and 3 units up



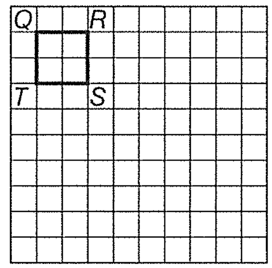
2. 4 units right and 1 unit up



3. 1 unit left and 2 units down

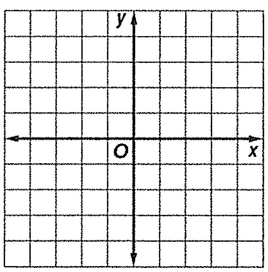


4. 5 units right and 3 units down

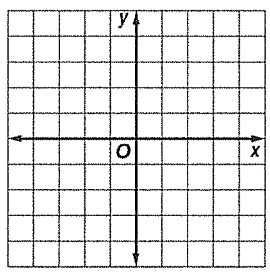


Graph the figure with the given vertices. Then graph the image of the figure after the indicated translation and write the coordinates of its vertices.

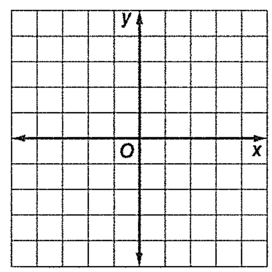
5. triangle *ABC* with vertices $A(-3, -1)$, $B(-4, -4)$, and $C(-1, -2)$ translated 4 units right and 1 unit up



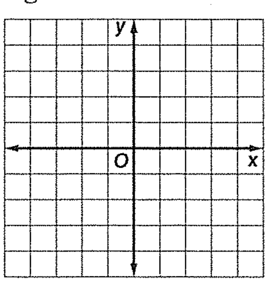
6. triangle *XYZ* with vertices $X(1, -2)$, $Y(3, -5)$, and $Z(4, 1)$ translated 5 units left and 3 units up



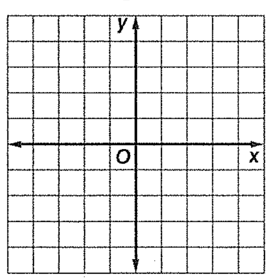
7. triangle *EFG* with vertices $E(1, 4)$, $F(-1, 1)$, and $G(2, -1)$ translated 3 units left and 1 unit down



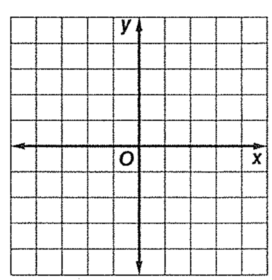
8. rhombus *WXYZ* with vertices $W(-4, 3)$, $X(-1, 1)$, $Y(2, 3)$, and $Z(-1, 5)$ translated 2 units right and 5 units down



9. rectangle *QRST* with vertices $Q(-2, -4)$, $R(-2, 1)$, $S(-4, 1)$, and $T(-4, -4)$ translated 3 units right and 3 units up



10. trapezoid *BCDE* with vertices $B(2, -1)$, $C(3, -3)$, $D(-3, -3)$, and $E(0, -1)$ translated 1 unit left and 4 units up



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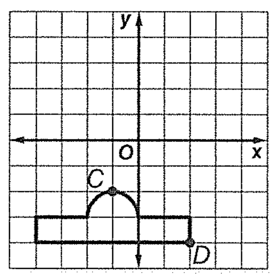
Problem-Solving Practice

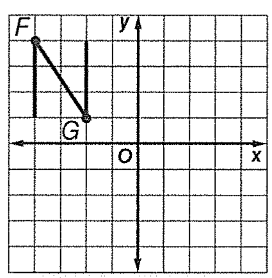
Translations

1. **BUILDINGS** The figure shows an outline of the White House in Washington, D.C., plotted on a coordinate system. Find the coordinates of points C and D after the figure is translated 2 units right and 3 units up.



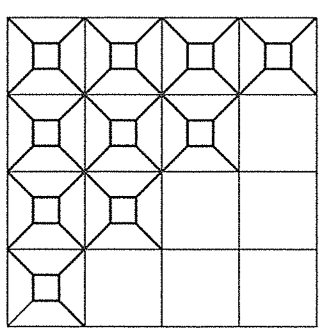
2. **BUILDINGS** Refer to the figure in Exercise 1. Find the coordinates of points C and D after the figure is translated 1 unit left and 4 units up.

3. **ALPHABET** The figure shows a capital "N" plotted on a coordinate system. Find the coordinates of points F and G after the figure is translated 2 units right and 2 units down.



4. **ALPHABET** Refer to the figure in Exercise 3. Find the coordinates of points F and G after the figure is translated 5 units right and 6 units down.

5. **QUILT** The beginning of a quilt is shown below. Look for a pattern in the quilt. Copy and translate the quilt square to finish the quilt.



6. **BEACH** Tylia is walking on the beach. Copy and translate her footprints to show her path in the sand.

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2.5 Identifying a Series and Determining Congruence or Similarity

Now that we understand how to create each type of transformation and how to combine with a series, we should be able to identify the specific transformations we see happening on the coordinate plane. There is no exact method for doing this other than intuition based on what we see.

A Series of Transformations

When identifying a series of transformation (or a single transformation), there is a three step process:

1.) Look for dilations
2.) Look for reflections and/or rotations
3.) Look for translations

Looking for Dilations

Check if there is a dilation first. Otherwise it can throw off your other transformations because the dilation moves the points closer to or farther away from the center of dilation. Ask the question, **"Has the picture been shrunk or enlarged?"** If so, try to decide by what scale factor. (For a series, we will assume the origin is the center of dilation.) One way to do this is find a side length that is easily measurable in the pre-image and check the corresponding side length on the image. If the pre-image has a side length of 2 units and the image has a side length of 1 unit, then the dilation had a scale factor of $c = \frac{1}{2}$. If the pre-image has a side length of 3 units and the image has a side length of 5 units, then the dilation had a scale factor of $c = \frac{5}{3}$.

Looking for Reflections/Rotations

The next question to ask yourself is, **"Has the picture orientation changed?"** If so, then a reflection or rotation was involved. Experiment to see which or if both have been used perhaps by using a piece of tissue paper to trace the pre-image. Next try turning, rotating, that tissue paper in 90° increments with it anchored at the origin. Remember that we're trying to find a way to get the orientations to match.

If turning the pre-image doesn't get the orientation of the image, try flipping the tissue paper over the axes. Try the x- and y-axis one at a time to get the orientation to match.

Looking for Translations

Finally check for a translation. Once you have the size and orientation matching, find the translation that you need to move the shape to the image. Remember that a translation takes into account both horizontal and vertical distance, so be sure to note both using the vector form.

Is the Series Unique?

That means, is the series that we find following this method the only series that will take the pre-image to the image? No. There may be other series that would work as well. Therefore if you are working and find a series that doesn't match the series of someone else or the answer document, that's OK. Just make sure that you can prove that your series works.

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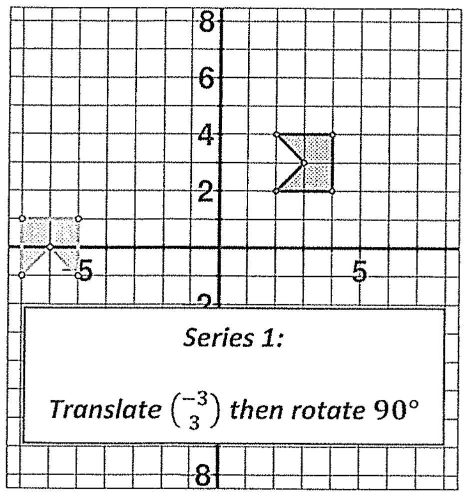
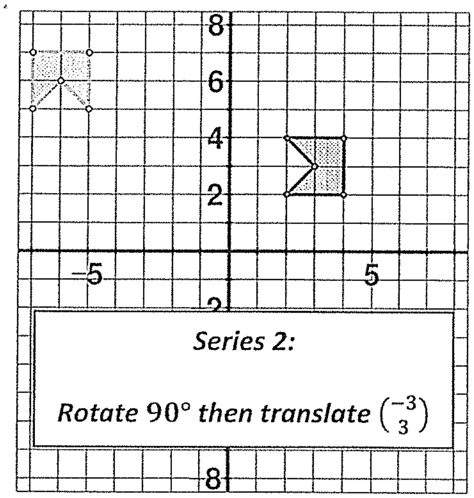
Does Order Matter?

For example, remember that a translation could occur before or after a reflection or rotation in a series. Each would produce a different result. Consider the following pictures (remember that the pre-image is darker in blue and the image is lighter in green). The first is a translation by the vector $\binom{-3}{3}$ followed by a 90° rotation. The second is a rotation by 90° followed by a translation by the vector $\binom{-3}{3}$. Notice that the two different series produce different results even though they use the same transformations. **The order matters!**

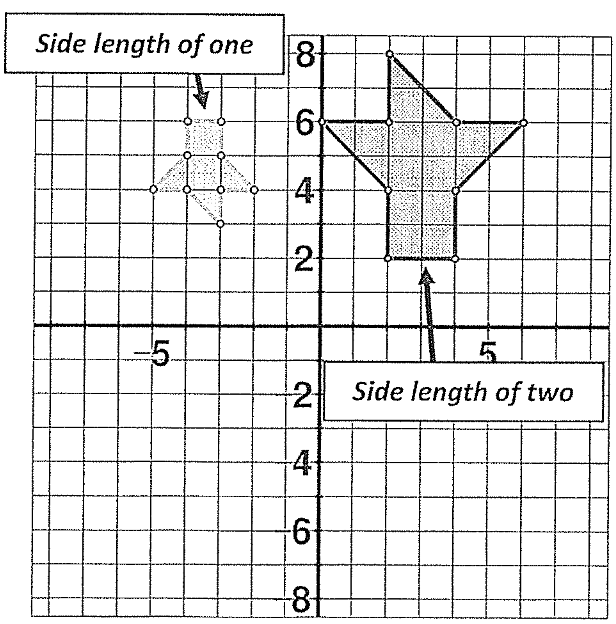




Consider again that there may be more than one series of transformations to take the pre-image to the image. For example, Series 1 could be a 90° rotation followed by a translation by vector $\binom{-3}{-3}$. That's just one example that there are often multiple ways to get a series.

All Together Now!

Now that we know the three step process (dilations, then reflections/rotations, then translations), let's work through an example problem.



Step 1: Dilation? Yes, this has clearly been dilated at some point because the image has been shrunk. Notice that the bottom of the pre-image has a length of two units and the corresponding side on the image has a length of one unit. So it has been cut in half. That means there was a dilation by a scale factor of $c = \frac{1}{2}$ with the origin as the center of dilation.

Step 2: Reflection/Rotation? Yes, there had to be one of these transformations because the orientation has changed. Now we need to experiment to decide if there a flip (reflection) or a turn (rotation) involved. Remember that it's possible that both were used as well.

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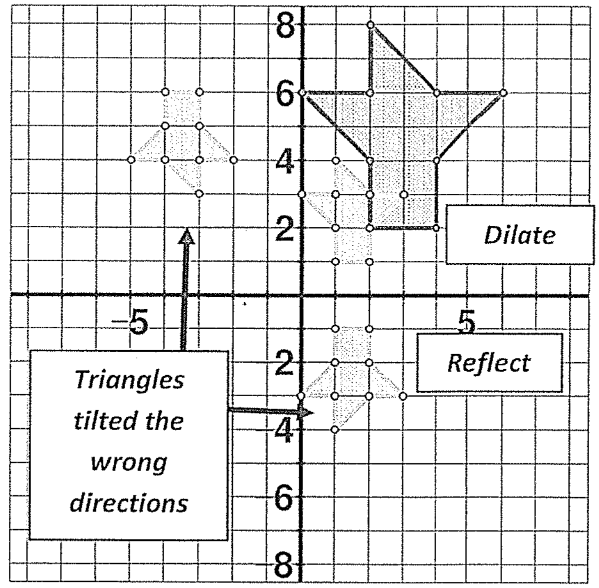
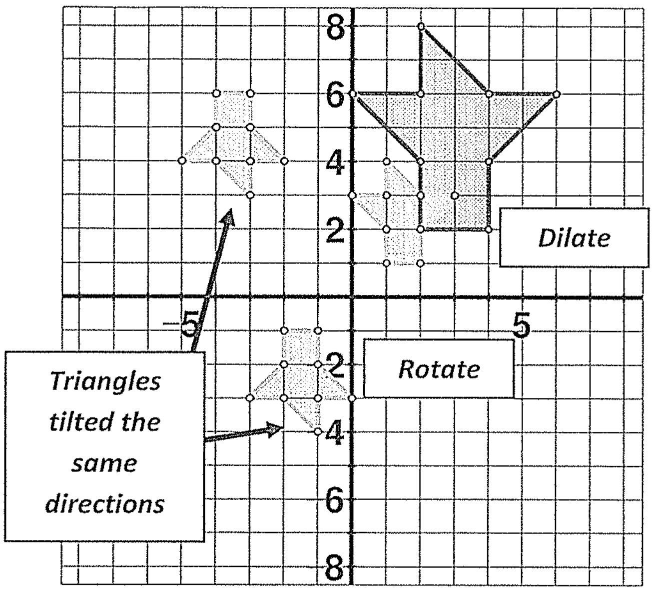
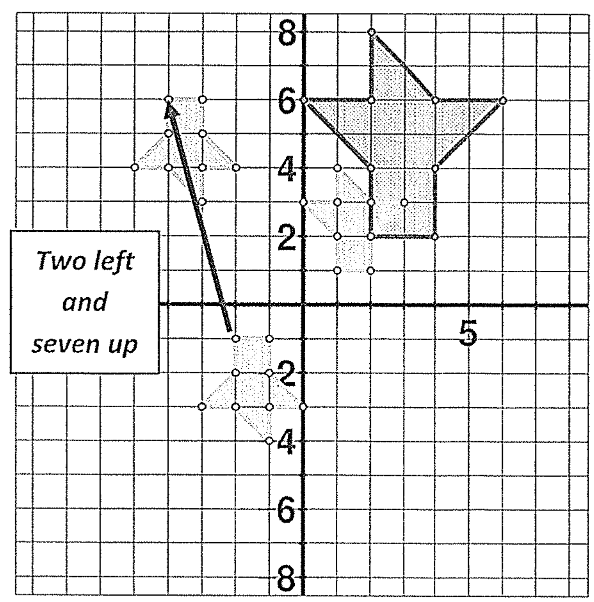
Just thinking and visualizing in our head a bit, a reflection across the y-axis won't work. The pointy ends would still be sticking up. So it might be a reflection across the x-axis. The picture to the right shows a dilation by $c = \frac{1}{2}$ and then a reflection across the x-axis. Does the orientation match that of the image? No. It's very close, but that doesn't do it because the middle triangle part is going the wrong direction.





Since reflections didn't work, we should try some rotations to match up the orientation. Picturing some rotations in our head, we should see that a quarter turn (90° rotation) won't work. That would lay the "tree" on its side instead of upside down. In the same way a three-quarters turn (270° rotation) won't work. So take a look at a dilation by $c = \frac{1}{2}$ followed by a 180° rotation at the left. Does the orientation now match that of the image? Yes. So the rotation by 180° was what we needed.

Step 3: Translation? Now we know the first two transformations in the series, but it still doesn't line up exactly with the image. That means we need to slide (translate) it to where the image is. Pick one set of corresponding points (one point from the picture we currently have after dilation and rotation and the other from the image) and check how we need to move that single point. That translation vector will work for the whole picture now that we have the size and orientation correct. Note that this translation vector is two left and seven up or $\binom{-2}{7}$.



Complete Series: So the complete series of transformations is a dilation by $c = \frac{1}{2}$, rotation of 180°, and translation by vector $\binom{-2}{7}$.

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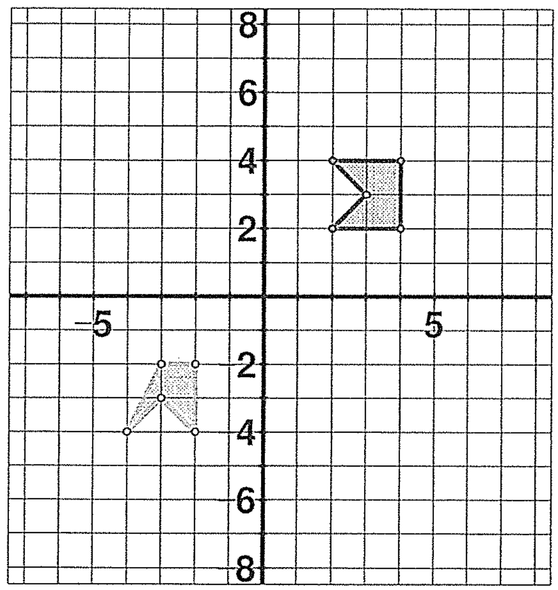
Congruent or Similar?

Just as with all previous examples, we can add a translation to a series of transformations to make more interesting pictures. Keep in mind that if the series contains only reflections, rotations and/or translations, the image will be congruent to the pre-image. If we use a dilation at any point, we will end up with a similar, not congruent, image. You will be expected to accurately identify whether the image is congruent or similar to the pre-image.

A third option does exist: the image and pre-image could be *neither* congruent nor similar. Consider a picture where the pre-image is Kermit the Frog and the image is Miss Piggy. Clearly those are neither congruent nor similar. No geometric transformation could turn Kermit into Miss Piggy. Consider the following example on the coordinate plane as a case where they are neither congruent nor similar.



Can you tell that the image is neither congruent nor similar to the pre-image? Yes, the two are not the same shape. One point seems to be out of place. The point $(-3, -2)$ on the image should be at $(-4, -2)$. That would make them congruent. In reality the image and pre-image are not really a true pre-image and image because no transformation will take one to the other. These shapes are neither congruent nor similar.

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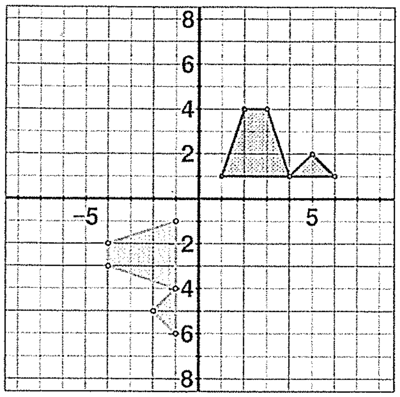
Lesson 2.5

Determine the specific series of transformations that took the pre-image (darker in blue) to the image (lighter in green). Be sure to give the specific vector, rotation angle, line of reflection and/or scale factor. Centers of dilation and rotation will be the origin. Then determine if the pre-image and image are similar or congruent.

1.



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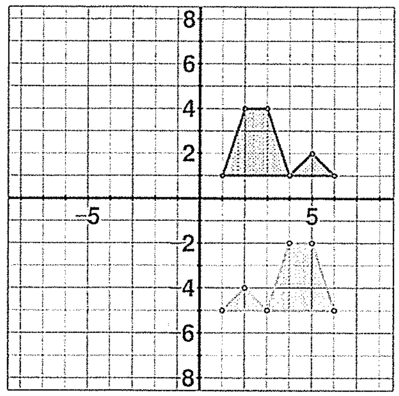

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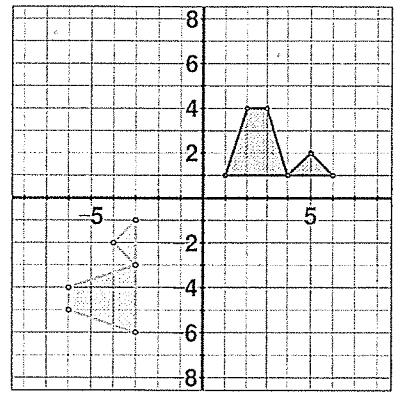

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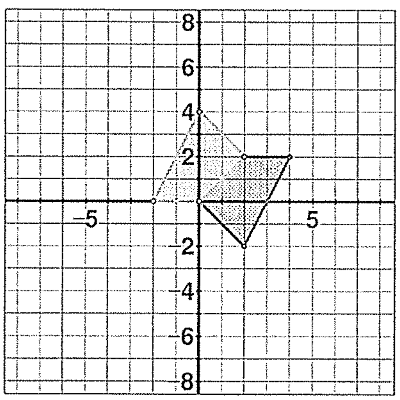

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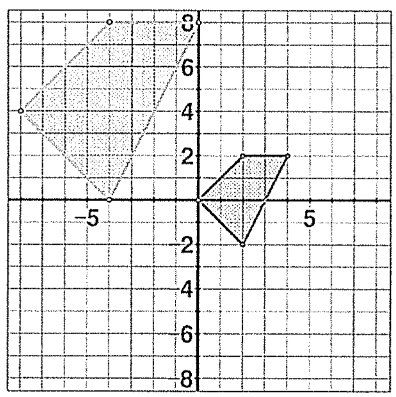

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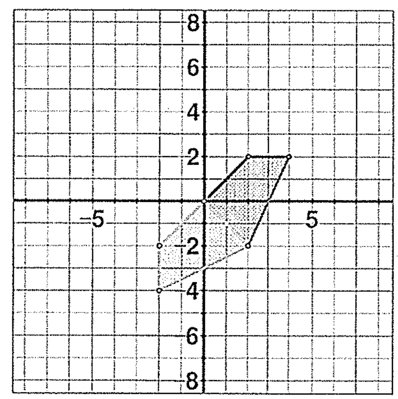

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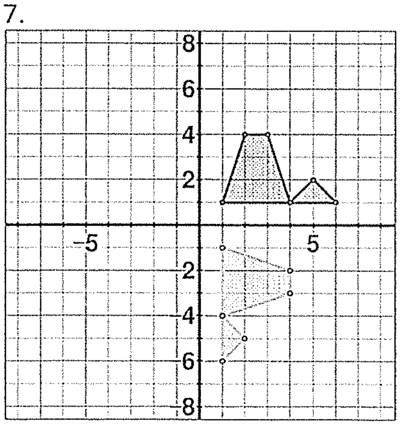

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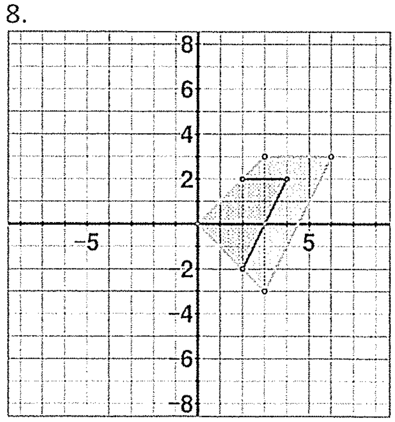

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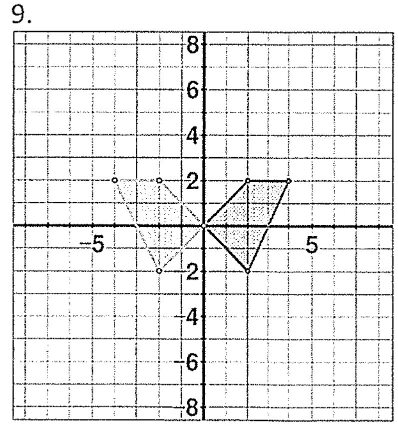

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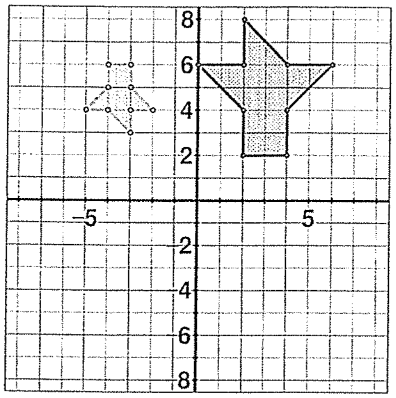

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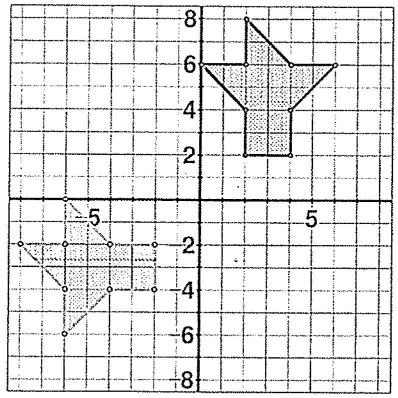

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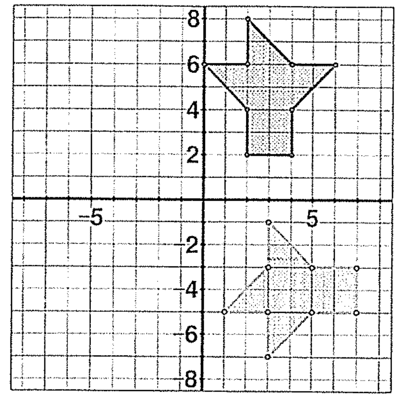

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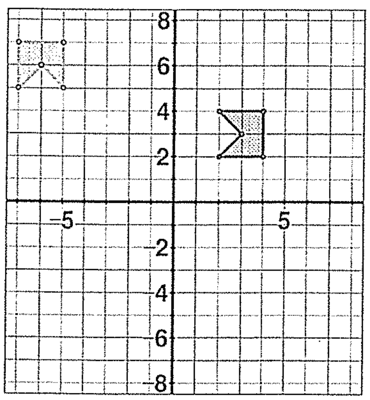

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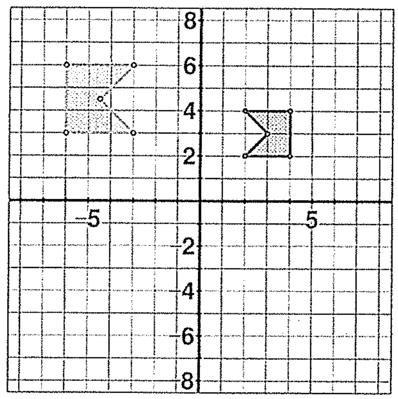

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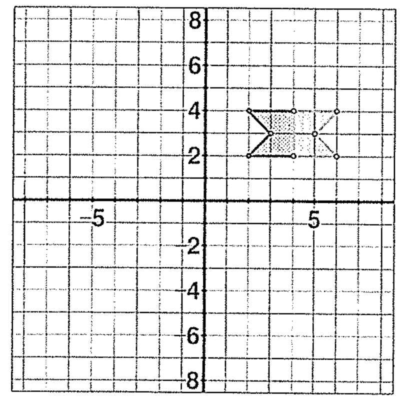

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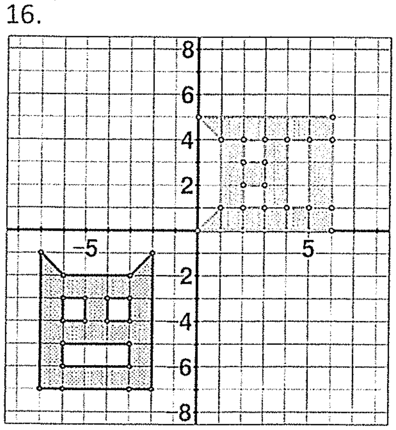

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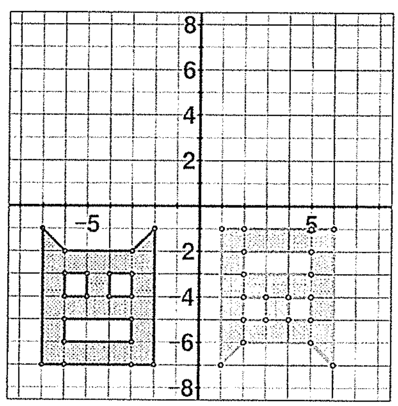

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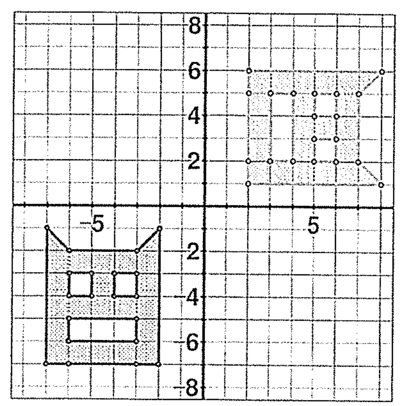

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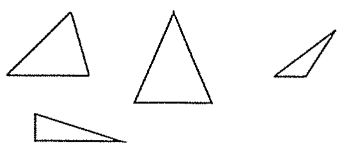
2.6 Sum of the Angles of a Triangle

There are other ways to determine if shapes are similar besides using transformations. We will explore one of these ways using triangles. To do so, we'll need to learn a few things about triangles in general before moving into the similar triangles.

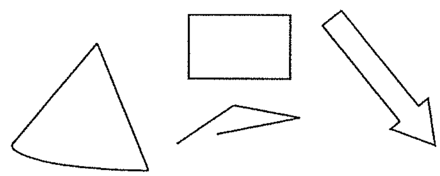
Reviewing the Triangle Definition

Let's look at some examples of triangles and non-triangles to find a definition of a triangle.

Triangles



Non-Triangles

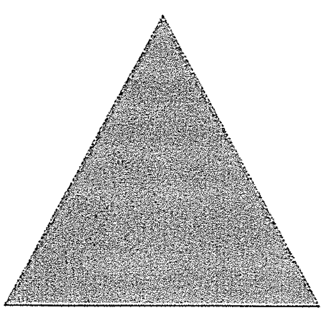


Take a moment to write down a definition of a triangle based on what you see above.

A **triangle** is a closed, three-sided figure where each side is a line segment. Closed means that the figure has no gaps in its perimeter, and requiring line segments to be the sides means that there won't be a curve of any kind.

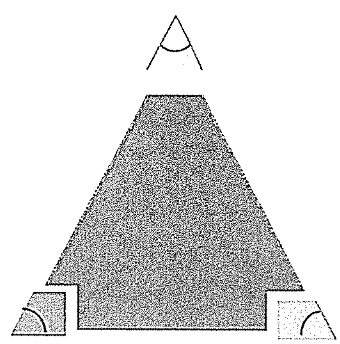
The Sum of the Angles of a Triangle

Now we can look at the angles of a triangle. Take a look at the following images:

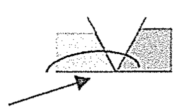


The shape to the left is a basic triangle. Now examine the shape to the right and see that we have cut off each of the angles. (As a side note, we should recall that each of these angles are called acute angles because they are less than 90°.)

Now look what happens if we stick all those angles together. What total angle measurement do we get?



What is the overall angle measurement?



The three angles put together made a straight angle, which measure 180°. As an experiment on your own, cut out a triangle from a sheet of paper. It can be any kind of triangle such as right scalene, obtuse isosceles, acute scalene, etc. Now rip off each angle (corner) of the triangle and put the vertices of the angles together to form one big angle. Does it make a straight angle as well?

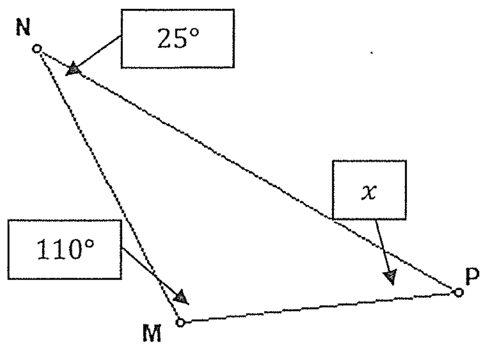
In fact, the angles of a triangle will always add up to 180°. This is sometimes called the Triangle Sum Theorem and allows us to find the missing angle measurements in a triangle.

Finding Missing Angles

Since the angles of a triangle (let's call them ∠1, ∠2, and ∠3) add up to 180°, we can write an equation to represent this.

$$m\angle 1 + m\angle 2 + m\angle 3 = 180°$$

Remember that the little m in front of the angle symbol means "the measure of". This means if we know the measurement of two of the angles of a triangle, we can solve for the third. Let's look at an example.



In order to solve for x, we substitute what we know into the equation. In this case we know that $m\angle M = 110°$, $m\angle N = 25°$, and we're missing the measure of $\angle P$. Since they all add up to 180°, we get the following equation:

$$x + 110° + 25° = 180°$$

$$x + 135° = 180°$$

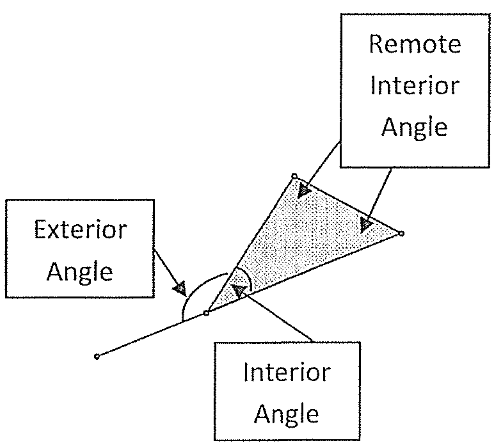
$$x + 135° - 135° = 180° - 135°$$

$$x = 45°$$

So the missing angle measurement must be 45° in order for all the angles to add up to 180°.

Exterior Angles

If we look at any angle in a triangle (we call it the **interior angle**) and extend one of the line segments beyond the angle, we get the **exterior angle**. The **remote interior** angles are the other two angles in the triangle. Looking at the picture below, what do we know about the exterior angle?



The exterior angle measurement plus the interior angle measurement together make a straight angle. That means that they should add up to 180°. So if the interior angle measure is 35°, then the exterior angle must measure 145°.

Also note that the exterior angles must be equal to the sum of the remote interior angles. Why? Since all the interior angle and remote interior angles add up to 180° and the interior angle and the exterior angle add up to 180°.

This means we can find the angle measurement of an exterior angle given the interior angle and vice versa.

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Lesson 2.6

Solve for the variable.

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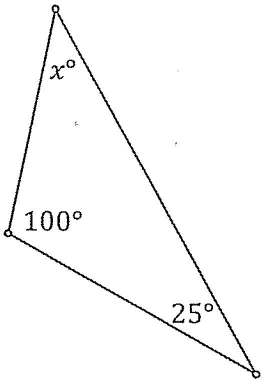

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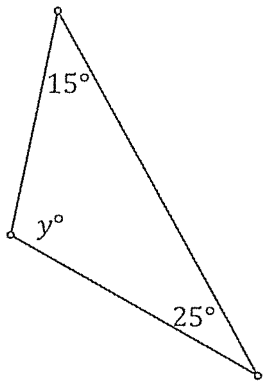

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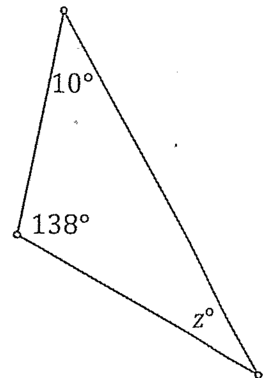

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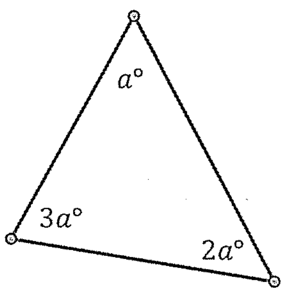

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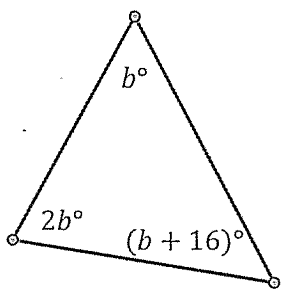

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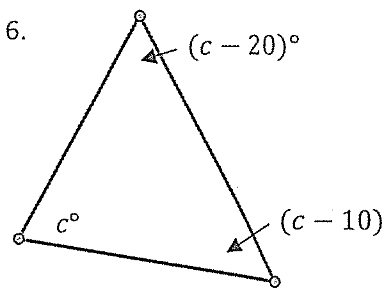

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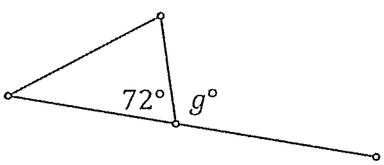

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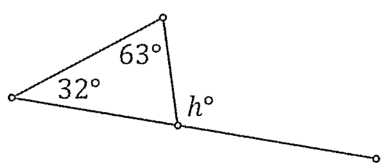

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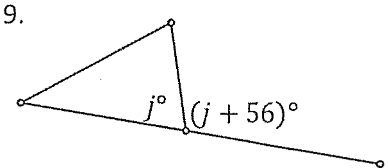

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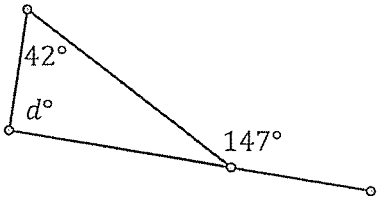

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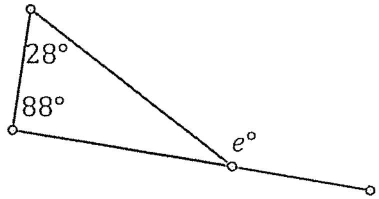

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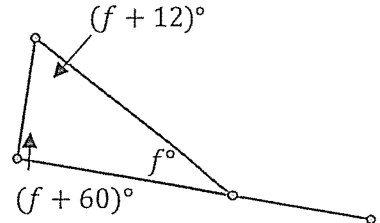

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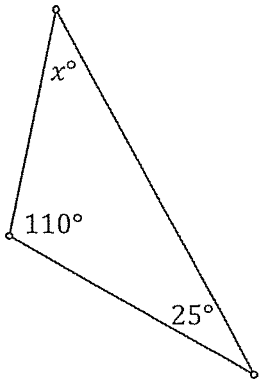

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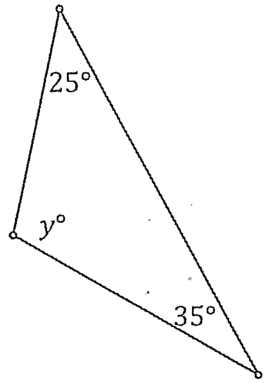

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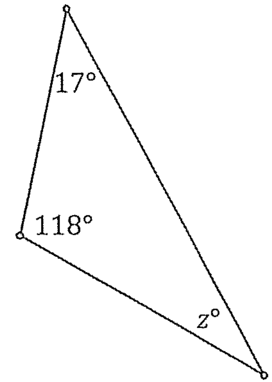

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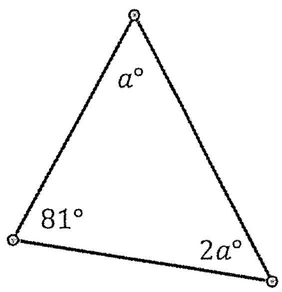

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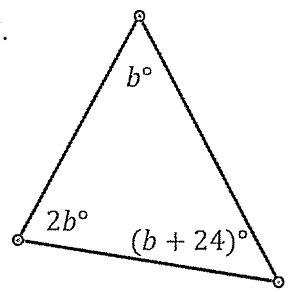

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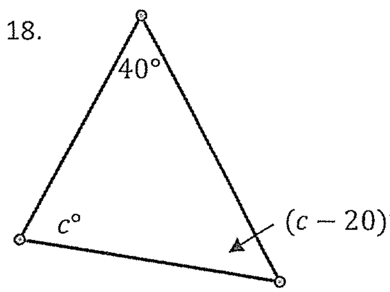

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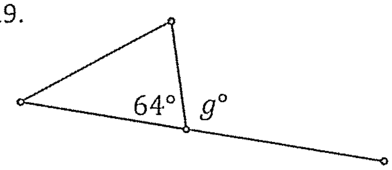

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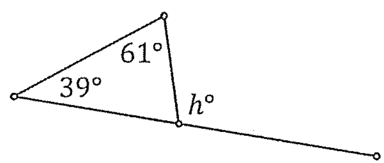

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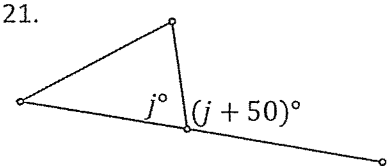

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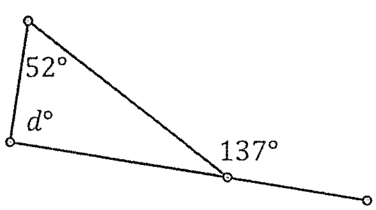

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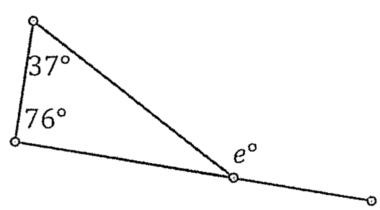

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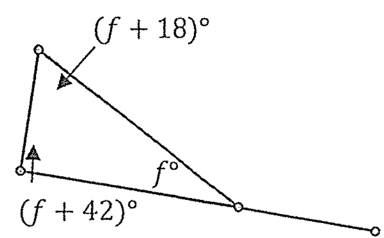

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NAME ______________________________ DATE ______________ PERIOD ________

Reteach

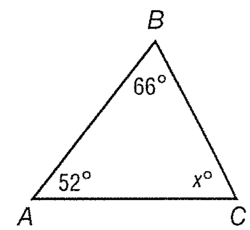
Angles of Triangles

- A **triangle** is formed by three line segments that intersect only at their endpoints.
- A point where the segments intersect is a **vertex** of the triangle.
- Every triangle also has three angles. The sum of the measures of the angles is 180°.

Example 1

Find the value of x in $\triangle ABC$.

$x + 66 + 52 = 180$	The sum of the measures is 180.
$x + 118 = 180$	Simplify.
$-118 \quad -118$	Subtract 118 from each side.
$x = 62$	

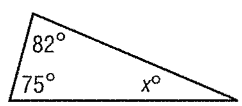


The value of x is 62.

Exercises

Find the the value of x in each triangle.

1.



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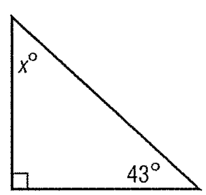

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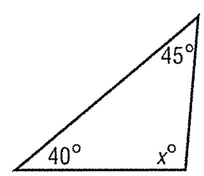

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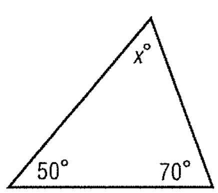

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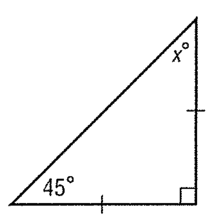

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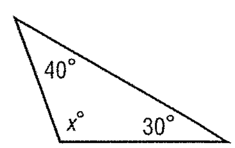

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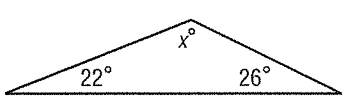

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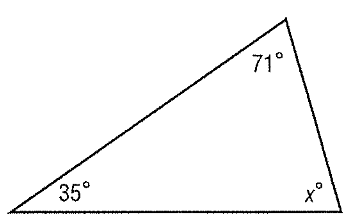

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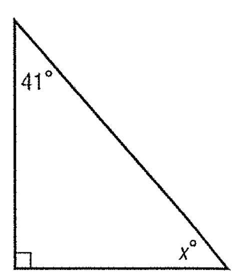

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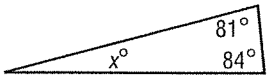
NAME ______________________ DATE ____________ PERIOD _______

Skills Practice

Angles of Triangles

Find the value of x in each triangle with the given angle measures.

1.



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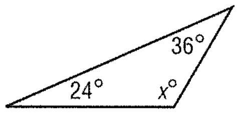

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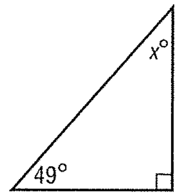

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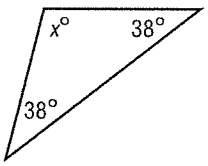

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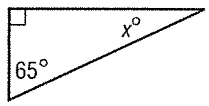

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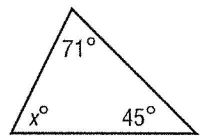


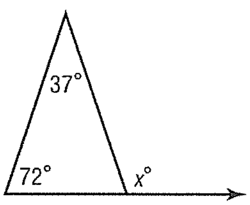
7. $57°, 51°, x°$

8. $x°, 126°, 22°$

9. $90°, x°, 50°$

Find the value of x in each triangle.

10.



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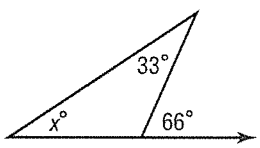

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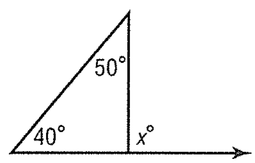

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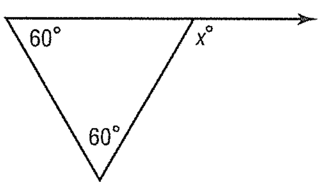

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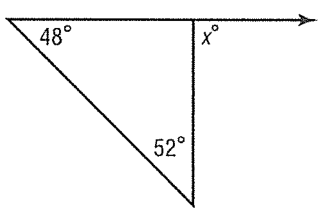

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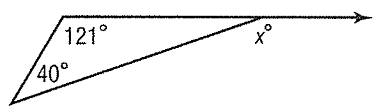

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NAME ______________________ DATE ____________ PERIOD ________

Problem-Solving Practice

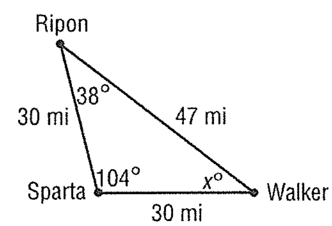
Angles of Triangles

1. **TAILORING** Each lapel on a suit jacket is in the shape of a triangle. Two of the three angles of each triangle measure 47° and 68°. What is the measure of the third angle?

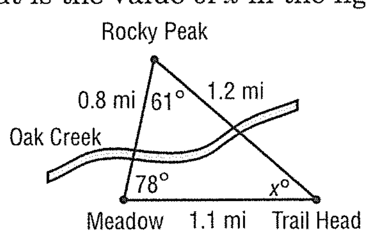
2. **FLAGS** A naval distress signal flag is in the shape of a triangle. Two of the three angles measure 55° each. What is the measure of the third angle?

3. **CARPENTRY** The supports of a wooden table are in the shape of a triangle. Find the angles of the triangle if the measures of the angles are in the ratio $4x : 4x : 10x$.

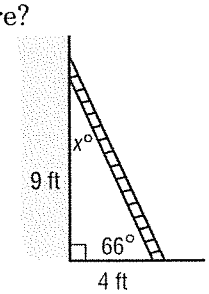
4. **MAPS** The three towns of Ripon, Sparta, and Walker form a triangle as shown below. What is the value of x in the triangle?



5. **HIKING** The figure shows the Oak Creek trail, which is shaped like a triangle. What is the value of x in the figure?



6. **LADDER** The figure shows a ladder leaning against a wall, forming a triangle. What is the value of x in the figure?



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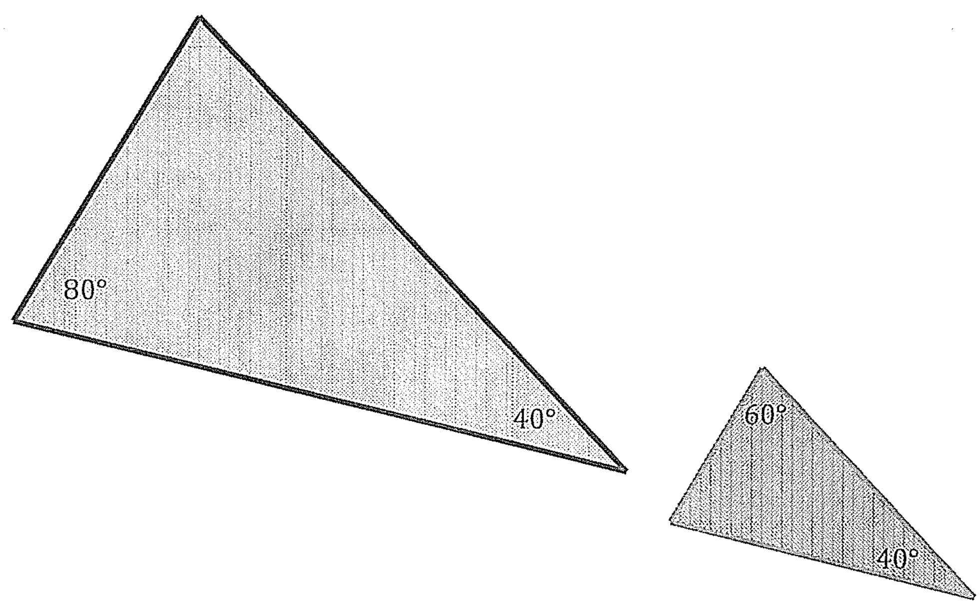
2.7 Similar Triangles

We have already talked about similar triangles when discussing proportions noting that similar triangles are triangles where the side lengths are proportional. Another way to think about similar triangles is that they are triangles of the same shape, but not necessarily the same size. Congruent triangles would have both the same size and shape.

We have also talked about similar triangles in the context of transformations in that triangles would be similar if there exists a set of translations, rotations, reflections, and/or dilations that take one shape to the other. Now we will examine similarity through examining their angles.

Angle-Angle Criterion for Similarity

We know that the angles of a triangle must add up to 180°. This means that if a triangle has two angle measurements of 40° and 80°, then the third angle must be 60°. Now if a second triangle has two angle measurements of 40° and 60°, we know the third angle must be 80°. This means the two triangles are the same shape, but not necessarily the same size. Alternately we may think of one as a dilation of the other. Either way we know that the triangles are similar. We call this the angle-angle criterion for similarity.



Lesson 2.7

Decide if the following triangles are similar and explain why using the angle-angle criterion.

1. Triangle 1 – $m\angle 1 = 45°, m\angle 2 = 45°$
   Triangle 2 – $m\angle 1 = 45°, m\angle 2 = 90°$

2. Triangle 1 – $m\angle 1 = 75°, m\angle 2 = 65°$
   Triangle 2 – $m\angle 1 = 65°, m\angle 2 = 140°$

3. Triangle 1 – $m\angle 1 = 50°, m\angle 2 = 30°$
   Triangle 2 – $m\angle 1 = 30°, m\angle 2 = 100°$

4. Triangle 1 – $m\angle 1 = 80°, m\angle 2 = 20°$
   Triangle 2 – $m\angle 1 = 80°, m\angle 2 = 80°$

5. Triangle 1 – $m\angle 1 = 60°, m\angle 2 = 20°$
   Triangle 2 – $m\angle 1 = 40°, m\angle 2 = 100°$

6. Triangle 1 – $m\angle 1 = 45°, m\angle 2 = 30°$
   Triangle 2 – $m\angle 1 = 30°, m\angle 2 = 100°$

7. Triangle 1 – $m\angle 1 = 40°, m\angle 2 = 30°$
   Triangle 2 – $m\angle 1 = 90°, m\angle 2 = 30°$

8. Triangle 1 – $m\angle 1 = 80°, m\angle 2 = 40°$
   Triangle 2 – $m\angle 1 = 40°, m\angle 2 = 60°$

9. Triangle 1 – $m\angle 1 = 35°, m\angle 2 = 95°$
   Triangle 2 – $m\angle 1 = 35°, m\angle 2 = 40°$

10. Triangle 1 – $m\angle 1 = 105°, m\angle 2 = 35°$
    Triangle 2 – $m\angle 1 = 40°, m\angle 2 = 105°$

11. Triangle 1 – $m\angle 1 = 35°, m\angle 2 = 95°$
    Triangle 2 – $m\angle 1 = 35°, m\angle 2 = 50°$

12. Triangle 1 – $m\angle 1 = 50°, m\angle 2 = 50°$
    Triangle 2 – $m\angle 1 = 50°, m\angle 2 = 90°$

13. Triangle 1 – $m\angle 1 = 25°, m\angle 2 = 115°$
    Triangle 2 – $m\angle 1 = 25°, m\angle 2 = 40°$

14. Triangle 1 – $m\angle 1 = 70°, m\angle 2 = 45°$
    Triangle 2 – $m\angle 1 = 45°, m\angle 2 = 65°$

15. Triangle 1 – $m\angle 1 = 5°, m\angle 2 = 15°$
    Triangle 2 – $m\angle 1 = 120°, m\angle 2 = 15°$

16. Triangle 1 – $m\angle 1 = 90°, m\angle 2 = 20°$
    Triangle 2 – $m\angle 1 = 90°, m\angle 2 = 80°$

17. Triangle 1 – $m\angle 1 = 5°, m\angle 2 = 15°$
    Triangle 2 – $m\angle 1 = 160°, m\angle 2 = 15°$

18. Triangle 1 – $m\angle 1 = 80°, m\angle 2 = 30°$
    Triangle 2 – $m\angle 1 = 70°, m\angle 2 = 30°$

19. Triangle 1 – $m\angle 1 = 45°, m\angle 2 = 55°$
    Triangle 2 – $m\angle 1 = 55°, m\angle 2 = 90°$

20. Triangle 1 – $m\angle 1 = 72°, m\angle 2 = 23°$
    Triangle 2 – $m\angle 1 = 85°, m\angle 2 = 23°$

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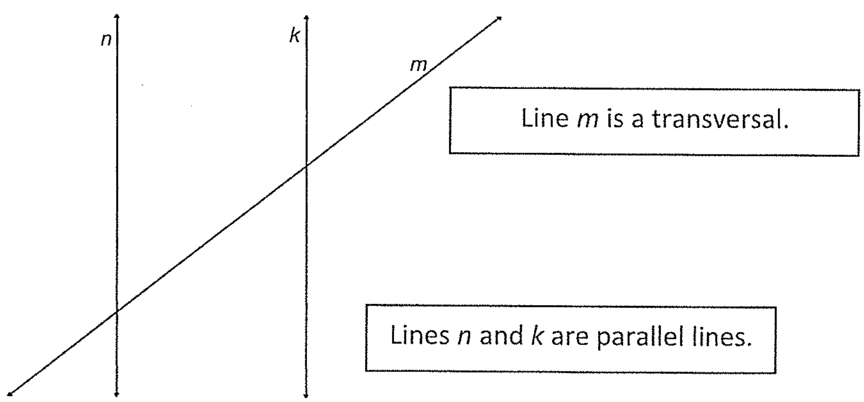
2.8 Parallel Lines Cut By A Transversal

When we extended the sides of triangles to find interior and exterior angle measurements, we created a situation that was very similar to parallel lines cut by a transversal. In this section we will explore what types of angles are created when we have a transversal cutting parallel lines.

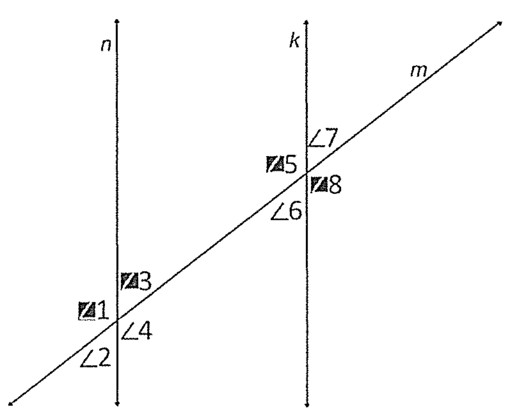
Reviewing Vocabulary

Parallel lines in two dimensions are lines that never cross or intersect. This means that the lines have the same orientation. If one line is going straight up and down, the parallel line will also be going straight up and down. For our purposes we will only look at parallel lines that are not overlapping. In other words, one line will not be sitting right on top of the other. Instead, our parallel lines will be more like railroad tracks.

A **transversal** is a line that intersects one or more parallel lines. This means that the transversal will have a different orientation from the parallel lines. So if the parallel lines are going straight up and down, then the transversal might be going left or right. The transversal could also be at some other angle (think of a positive 2 slope for example).



Notice that in this picture there are eight angles that are created. We typically name those angles using the numbers 1 through 8. It could look something like this.



You should notice right away that several of these angles look like they have the same angle measurement. In fact it looks like the four acute angles have equal measurement and the four obtuse angles have equal measurement. In fact this is the case, but let's examine why this is true and classify the different types of angles we find here.

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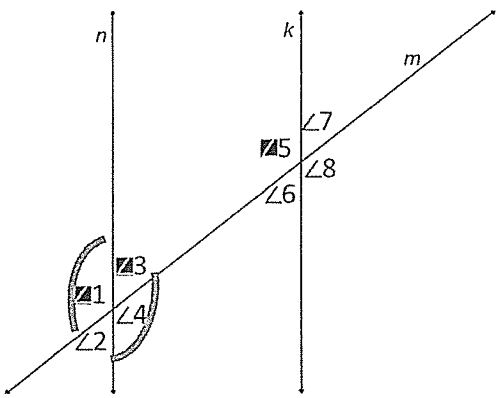
Vertical Angles

Note that $\angle 1$ and $\angle 3$ must add up to $180°$ because they sit on a line. They are like the exterior and interior angles of a triangle adding up to $180°$. However, the same argument can be made for $\angle 4$ and $\angle 3$. They must also add up to $180°$. Therefore we know that $\angle 4$ and $\angle 1$ must have the same measurement. In other words we know that $\angle 4 \cong \angle 1$ (pronounced "angle 4 is congruent to angle 1") or $m\angle 4 = m\angle 1$ (pronounced "the measure of angle 4 is equal to the measure of angle 1").



We call this type of congruent angle **vertical angles**. One way to remember vertical angles is to remember that they sit in a "V". Where are the other vertical angles in our picture?

The vertical angles come in pairs. A second pair of vertical angles is $\angle 2$ and $\angle 3$. A third pair of vertical angles is $\angle 5$ and $\angle 8$. The fourth pair of vertical angles is $\angle 6$ and $\angle 7$.

We can now ask questions such as: what is $m\angle 3$ if $m\angle 2 = 40°$? Since we know that they are vertical angles, they must be congruent. Therefore the answer is $m\angle 2 = 40°$.

Corresponding Angles

Now imagine taking the angles formed by line *n* and line *m* and sliding them up so that they overlap the angles formed by line *k* and line *m*. Now which angles do we know are congruent? In other words, angles 1 through 4 will be sitting right on top of angles 5 through 8. Which ones line up? These angles are called **corresponding angles** and are congruent.

For starters, you should notice that $\angle 1$ sits on top of $\angle 5$. So $\angle 1$ and $\angle 5$ are a pair of corresponding angles. A second pair would be $\angle 2$ and $\angle 6$. A third pair would be $\angle 3$ and $\angle 7$. The final pair would be $\angle 4$ and $\angle 8$. This means that if $m\angle 2 = 40°$ then $m\angle 6 = 40°$ must be true since they are corresponding angles. One way to remember corresponding angles is to think of the angles that are on the same corner. Cor-ner and Cor-responding angles.

Alternate Interior Angles

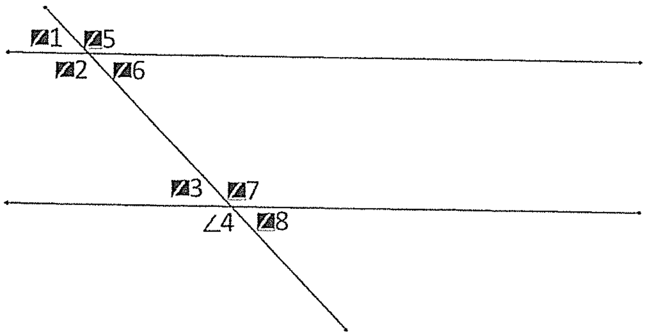
Using the same picture above, look at $\angle 3$ and $\angle 6$. These are called **alternate interior angles** and are also congruent. They are called alternate interior angles because they alternate which side of the transversal they are on ($\angle 3$ is on top of the transversal in this case and $\angle 6$ is on bottom) and because they are inside the parallel lines (hence the word "interior"). Alternate interior angles are also congruent. There are two pairs of this type of angle in our picture: $\angle 3$ and $\angle 6$ and then $\angle 4$ and $\angle 5$.

Alternate Exterior Angles

In a similar fashion, if the angles lie on alternate sides of the transversal and are outside the parallel lines, the angles are called **alternate exterior angles**. Alternate exterior angles are congruent as well. In our picture there are two pairs of alternate exterior angles which are $\angle 1$ and $\angle 8$ and then $\angle 2$ and $\angle 7$.

Lesson 2.8

Use the following picture to answer the questions.



1. Name a pair of vertical angles.

2. Name a pair of corresponding angles.

3. Name a pair of alternate interior angles.

4. Name a pair of alternate exterior angles.

5. If $m\angle 2 = 110°$, what is $m\angle 5$?

6. If $m\angle 2 = 110°$, what is $m\angle 4$?

7. If $m\angle 2 = 110°$, what is $m\angle 7$?

8. If $m\angle 1 = 70°$, what is $m\angle 8$?

9. If $m\angle 1 = 70°$, what is $m\angle 7$?

10. If $m\angle 2 = 140°$, what are the measures of all the other angles?

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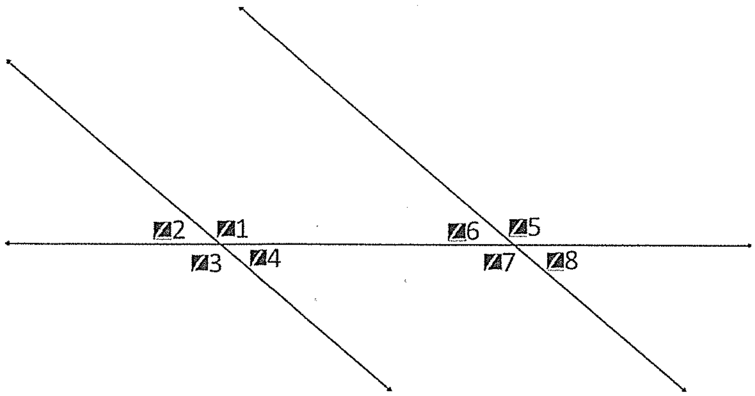

11. Name all the pairs of vertical angles.

12. Name all the pairs of corresponding angles.

13. Name all the pairs of alternate interior angles.

14. Name all the pairs of alternate exterior angles.

15. If $m\angle 2 = 40°$, what is $m\angle 8$?

16. If $m\angle 2 = 40°$, what is $m\angle 4$?

17. If $m\angle 1 = 140°$, what is $m\angle 7$?

18. If $m\angle 1 = 140°$, what is $m\angle 5$?

19. If $m\angle 1 = 140°$, what is $m\angle 6$?

20. If $m\angle 2 = 35°$, what are the measures of all the other angles?

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Review Unit 2: Congruence and Similarity

You may not use a calculator.

Unit 2 Goals

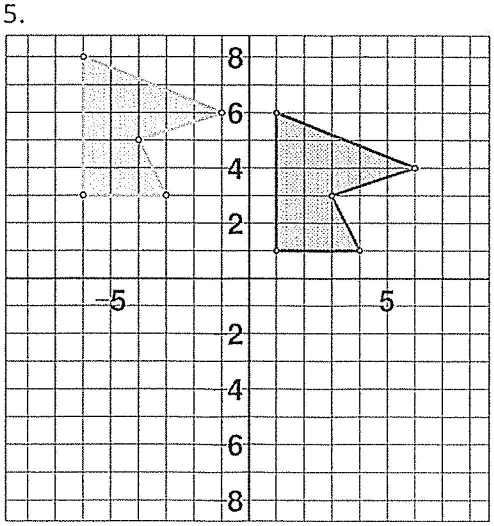
- *Verify experimentally the properties of rotations, reflections, and translations: a) Lines are taken to lines, and line segments to line segments of the same length; b) angles are taken to angles of the same measure; c) parallel lines are taken to parallel lines. (8.G.1)*
- *Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them. (8.G.2)*
- *Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates. (8.G.3)*
- *Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them. (8.G.4)*
- *Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. (8.G.5)*

Answers the following questions.

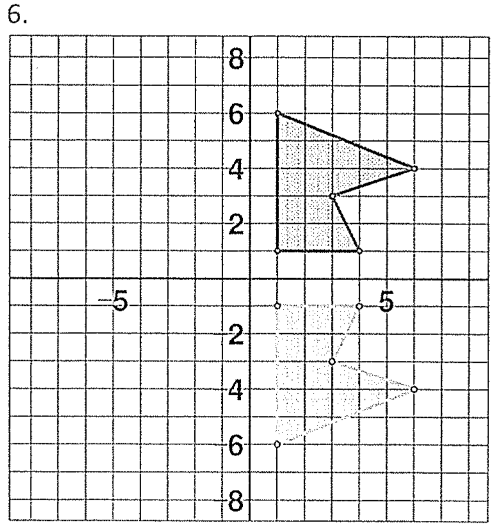
1. If you translate the pre-image A, will the image A' be congruent or only similar?
2. If you dilate the pre-image B, will the image B' be congruent or only similar?
3. If you rotate the pre-image C, will the image C' be congruent or only similar?
4. If you reflect the pre-image D, will the image D' be congruent or only similar?

Name the specific transformation shown in each picture as a translation, rotation, reflection, or dilation. Then determine if the pre-image (darker in blue) and the image (lighter in green) are similar or congruent.

5.



6.



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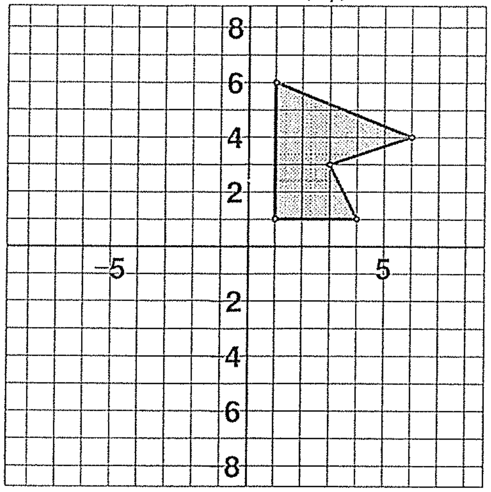
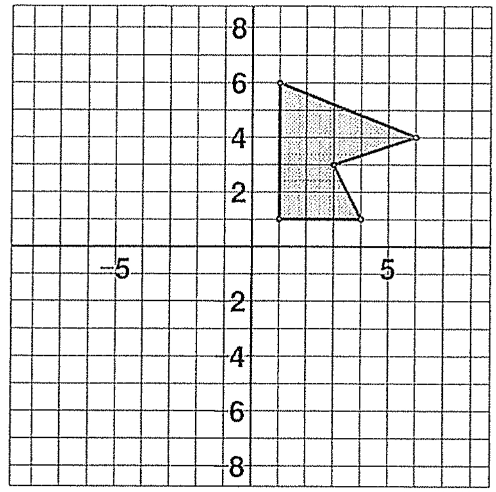
Apply the given transformation or series of transformations to the given pre-image.

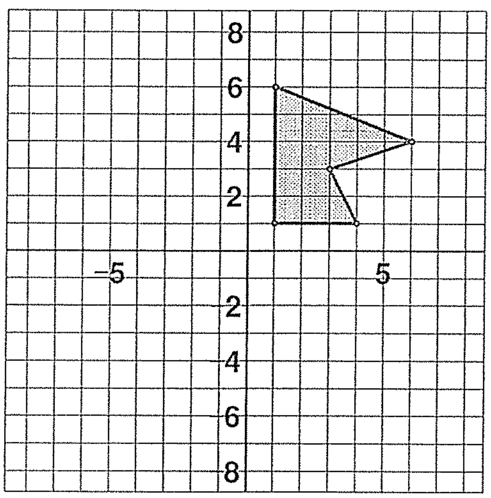
7. Translation by vector $\binom{-6}{-7}$



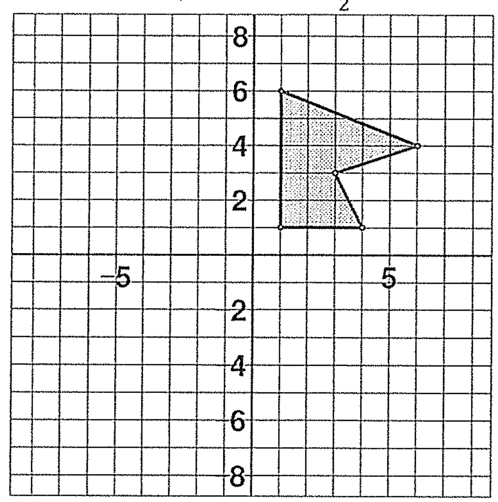
8. Rotation by 180°



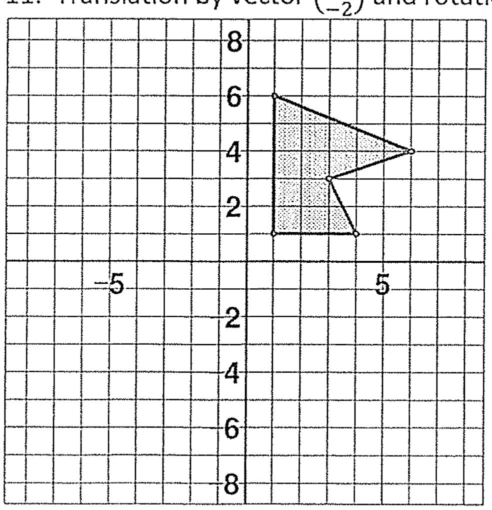
9. Reflection across the x-axis



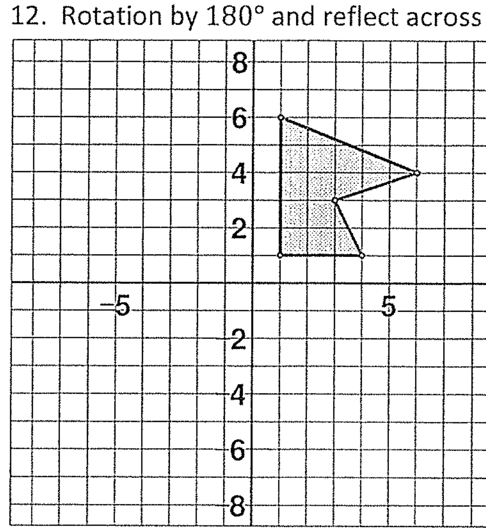
10. Dilation by scale factor $\frac{1}{2}$



11. Translation by vector $\binom{2}{-2}$ and rotation by 90°

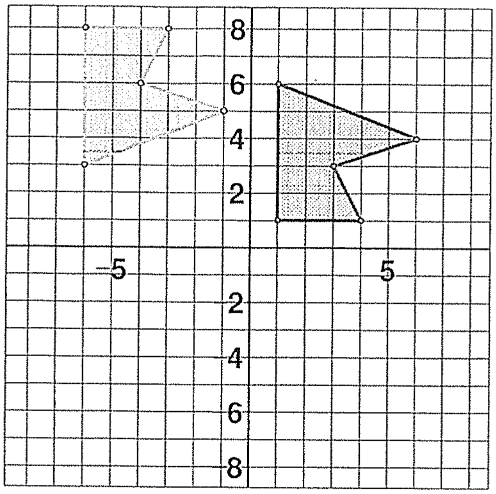


12. Rotation by 180° and reflect across y-axis



Identify a specific series of transformations that would take the pre-image (darker in blue) to the image (lighter in green). Then tell whether the pre-image and image are congruent or similar.

13.



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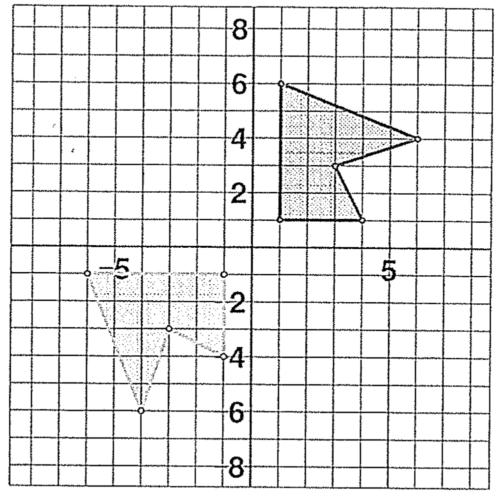
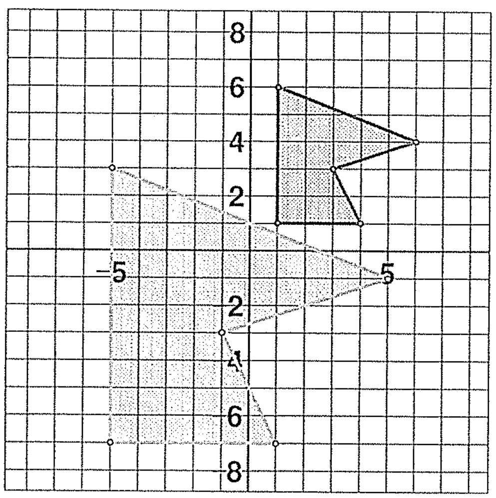


15.



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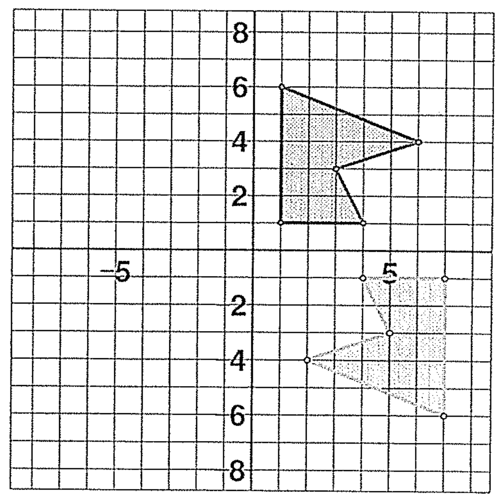
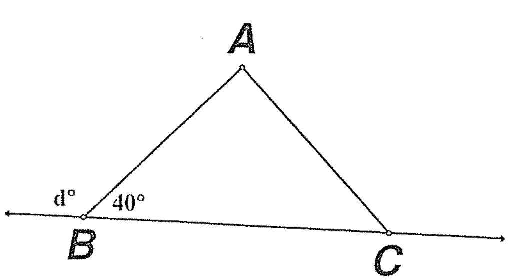


Find the angle measure of each missing angle.

17.



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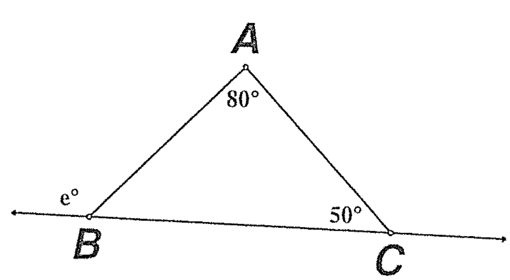

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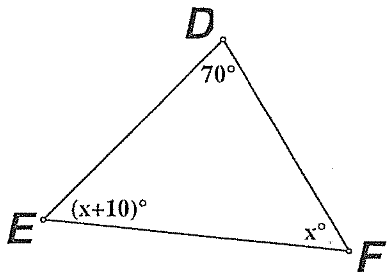
19.



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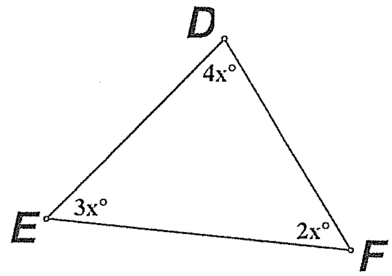
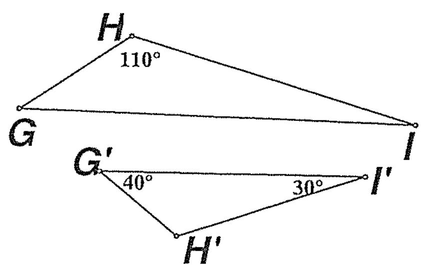
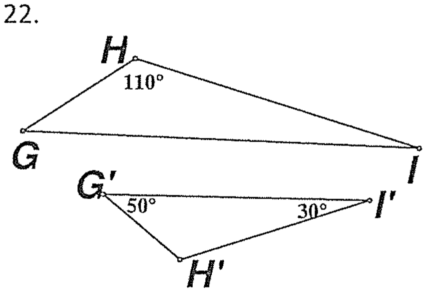


Determine if the following triangles are similar or not and explain why or why not.

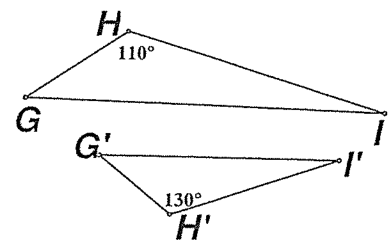
21.



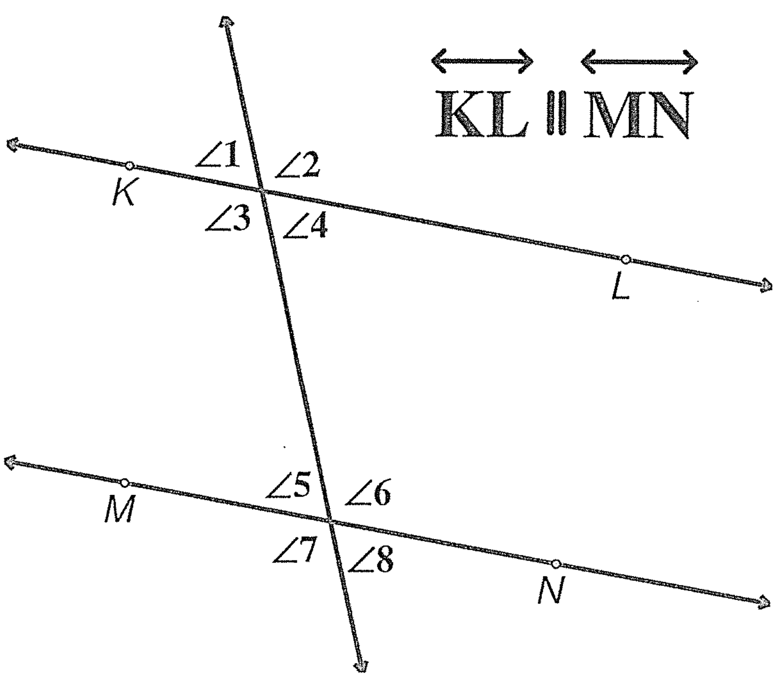
22.



23.



Use the picture to answer the following questions.



24. What type of angles are $\angle 1$ and $\angle 4$?

25. What type of angles are $\angle 1$ and $\angle 5$?

26. What type of angles are $\angle 1$ and $\angle 8$?

27. What type of angles are $\angle 3$ and $\angle 6$?

28. List all the angles congruent to $\angle 4$.

29. If $m\angle 1 = 75°$, what is $m\angle 8$?

30. If $m\angle 2 = 135°$, what is $m\angle 4$?

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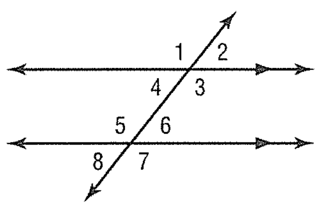
Reteach

Lines

- **Perpendicular lines** are lines that intersect at right angles.
- **Parallel lines** are two lines in a plane that never intersect or cross.
- A line that intersects two or more other lines is called a **transversal.**
- If the two lines cut by a transversal are parallel, then these special pairs of angles are congruent: ***alternate interior angles, alternate exterior angles, and corresponding angles.***

Example 1

Classify ∠4 and ∠8 as *alternate interior, alternate exterior,* or *corresponding.*



∠4 and ∠8 are in the same position in relation to the transversal on the two lines. They are corresponding angles.

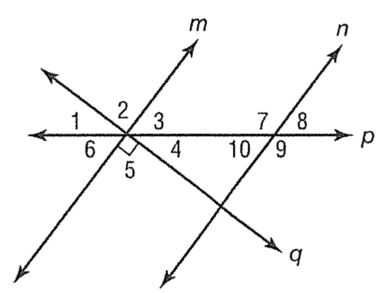
Example 2

Refer to the figure in Example 1. Find $m\angle 2$ if $m\angle 8 = 58°$.

Since ∠2 and ∠8 are alternate exterior angles, $m\angle 2 = 58°$

Exercises

In the figure at the right, line *m* and line *n* are parallel. If $m\angle 3 = 64°$, find each given angle measure. Justify each answer.



1. $m\angle 8$

2. $m\angle 10$

3. $m\angle 4$

4. $m\angle 6$

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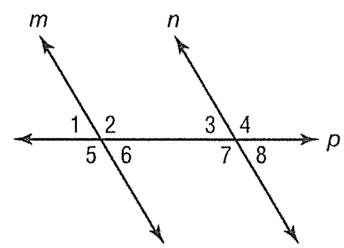
Skills Practice

Lines

For Exercises 1–12, use the figure at the right. In the figure, line *m* is parallel to line *n*.



Classify each pair of angles as *alternate interior, alternate exterior, or corresponding.*

1. $\angle 1$ and $\angle 8$

2. $\angle 5$ and $\angle 7$

3. $\angle 3$ and $\angle 6$

4. $\angle 2$ and $\angle 4$

5. $\angle 2$ and $\angle 7$

6. $\angle 4$ and $\angle 5$

If $m\angle 4 = 122°$, find each given angle measure. Justify your answer.

7. $m\angle 8$

8. $m\angle 5$

9. $m\angle 2$

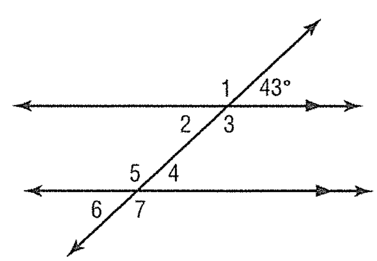
10. $m\angle 1$

11. $m\angle 6$

12. $m\angle 7$

For Exercises 13 and 14, use the figure at the right.

13. List all the angles congruent to the given angle. Explain your reasoning.



14. List all the angles congruent to $\angle 5$. Explain your reasoning.

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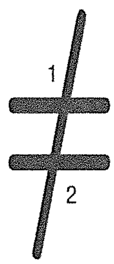
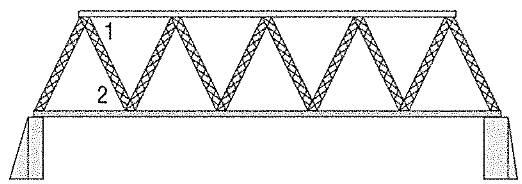
Problem-Solving Practice

Lines

1. SYMBOLS The symbol below is an equal sign with a slash through it. It is used to represent *not equal to* in math, as in $1 \neq 2$. If $m\angle 1 = 108°$, classify the relationship between $\angle 1$ and $\angle 2$. Then find $m\angle 2$. Assume the equal sign consists of parallel lines.

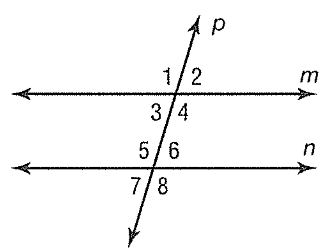


2. BRIDGE Arturo is designing a bridge for science class using parallel supports for the top and bottom beam. Find $m\angle 2$ if $m\angle 1 = 60°$.



3. LEG LIFTS For cheerleading practice, Kiara must be able to lift her legs so that they are parallel to her outstretched arms. For each side of her body, what is the relationship between the angle formed by her arms and the floor and the angle formed by her legs and the floor?

4. ALGEBRA In the figure, line m is parallel to line n. If $m\angle 3 = 7x - 10$ and $m\angle 6 = 5x + 10$, What is the measure of $\angle 3$ and $\angle 6$?



5. ALGEBRA Refer to the figure in Exercise 4. If $m\angle 1 = 4x + 40$, and $m\angle 5 = 120°$, what is the value of x?

6. ART The drawing below shows the side view of a drawing easel. The brace is parallel to the ground. If $m\angle A$ is 82°, what is the measure of $\angle B$?